# Contents

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D-branes are a very efficient tool to engineer gauge theories and study their properties. They have been instrumental in understanding the non-perturbative behavior of gauge theories, in studying their moduli space, and in uncovering new dualities. We give a pedagogical overview of some aspects of the brane constructions focusing on simple examples of theories with eight supercharges in three and four dimensions. Among other things, we discuss configurations based on brane within branes, orbifolds, branes ending on branes, and the Hanany-Witten construction. We discuss how the moduli space of instantons and monopoles are realized in terms of branes. We present various applications, including mirror symmetry in three dimensions, the Seiberg-Witten curve of four-dimensional theories, and an introduction to theories of class S.

D 膜是构造规范理论并研究其性质的非常高效的工具。它在理解规范理论的非微扰行为、研究规范理论的模空间以及发现新对偶方面发挥了关键作用。本文以教学风格概述膜构造的若干方面，重点关注三维和四维中具有八个超荷的理论的简单例子。此外，我们还讨论了基于膜中膜、orbifold(轨形)、终止于膜上的膜以及汉纳尼-威滕构造的构型。我们讨论了瞬子和磁单极的模空间如何用膜实现。我们给出了多种应用，包括三维镜像对称、四维理论的塞伯格-维滕曲线，以及 S 类理论的简介。

## Keywords

## 关键词

D-branes - Gauge theories - Supersymmetry

D 膜 - 规范理论 - 超对称

## Introduction

## 引言

The role that D-branes have played in understanding non-perturbative properties of supersymmetric gauge theories cannot be underestimated. In the last 25 years, branes have been used to understand the moduli space of gauge theories at the non-perturbative level, to study their infrared (IR) limit, and to discover new fixed points and new dualities. In these notes, we give a pedagogical introduction to the main ingredients in the brane realization of gauge theories. For reasons of space, we focus on theories with 8-16 supercharges in three and four dimensions. These vintage examples, where the brane description works at its best, contain all the main ideas behind the D-brane construction for gauge theories.

D 膜在理解超对称规范理论的非微扰性质中发挥的作用怎么强调都不为过。过去 25 年间，人们利用膜从非微扰层面研究了规范理论的模空间，分析了它们的红外 (IR) 极限，还发现了新的不动点与新的对偶性。本讲义将教学式地介绍规范理论膜构造的核心要素。受篇幅所限，我们聚焦三维与四维中含有 8 到 16 个超荷的理论。这些经典例子是膜描述最适用的场景，涵盖了规范理论 D 膜构造背后的所有核心思想。

We start by discussing how to realize gauge theories with maximal or half-maximal supersymmetry on the worldvolumes of D-branes in type II string theory. The main methods involve D-branes with codimension four, D-branes at orbifold singularities, and D-branes ending on other branes (the Hanany-Witten construction). We discuss in detail the moduli space of such configurations, which is often quantum corrected. With the help of various simple examples, we demonstrate that

我们首先讨论，如何在 II 型弦论中 D 膜的世界体积上实现最大或半最大超对称的规范理论。主要方法包括余维数为四的 D 膜、处于轨形奇点的 D 膜，以及端点落在其他膜上的 D 膜 (即 Hanany-Witten 构造)。我们详细讨论了这类构型的模空间，该空间通常存在量子修正。通过多个简单的例子我们可以说明:

- The full quantum corrected moduli space can be often inferred by the string embedding. This is obtained by considering the branes as probes of the spacetime geometry.

- 完整的经量子修正的模空间通常可以通过弦嵌入推导得到，这是通过将膜视为时空几何的探针得到的。

- Many interesting moduli spaces of geometrical interest (monopoles, instantons, ALE spaces, etc.) have an interpretation in terms of branes. This construction recovers and extends some classical mathematical constructions.

- 许多具有几何意义的重要模空间 (磁单极子、瞬子、ALE 空间等) 都可以用膜给出解释，该构造重现并推广了一些经典的数学构造。

- Using string dualities, we can map various brane configurations among themselves, uncovering and explaining IR dualities between gauge theories.

- 利用弦对偶性，我们可以将不同的膜构型相互映射，从而揭示并解释规范理论之间的红外对偶性。

We discuss some applications of the previous techniques. In particular, using the Hanany-Witten construction [1], we discuss in some details mirror symmetry [2] for theories with eight supercharges in three dimensions and the Seiberg-Witten curve [3,4] for theories with eight supercharges in four dimensions. We conclude with an introduction to superconformal class S theories in four dimensions [5,6].

我们讨论了上述技术的一些应用。具体来说，利用 Hanany-Witten 构造 [1]，我们详细讨论了三维八超荷理论的镜像对称性 [2]，以及四维八超荷理论的西伯格-维滕曲线 [3,4]。最后我们介绍了四维的 S 类超共形理论 [5,6]。

Many fundamental topics are necessarily missing. We do not discuss brane realizations for five- and six-dimensional gauge theories, the brane construction for theories with four supercharges in different dimensions, the geometrical engineering of gauge theories using singularities in the internal space, and the AdS/CFT correspondence. We do not also discuss the connection of supersymmetric indices and partition functions, tools that have been proven to be very useful for understanding moduli spaces and dualities.

囿于篇幅，许多基础课题不得不省略。我们不讨论五维和六维规范理论的膜实现，不讨论不同维度下四超荷理论的膜构造，不讨论利用内空间奇点进行规范理论的几何工程，也不讨论 AdS/CFT 对应。我们同样没有讨论超对称指标与配分函数的相关联系，而这些工具已被证明对研究模空间和对偶性非常有用。

## A Brief Introduction to the D-Brane Action

## D 膜作用量简介

In this section, we briefly review the main properties of D-branes and their coupling to space-time fields that will be used in the following. We will use Polchinski's book notations [7], to which we refer for the necessary background about D-branes and string dualities. The reader can also consult the chapter of this handbook specifically devoted to D-branes.

本节我们将简要回顾后续会用到的 D 膜主要性质及其与时空场的耦合。我们采用 Polchinski 著作 [7] 的记号，有关 D 膜与弦对偶的必要背景知识可查阅该著作，读者也可参考本手册中专门介绍 D 膜的章节。

Let us focus on type II string theory. A $p$ -brane is an extended object charged under the $(p+1)$ -form Ramond-Ramond (R-R) gauge field $C_{p+1}$ . The action of a $p$ -brane is a straightforward generalization of the action of a particle of mass $m$ and charge $q$ under a classical field $A_\mu$ , which is given by

我们以 II 型弦理论为例展开讨论。$p$ 膜是一类延展物体，携带 $(p+1)$ 型 Ramond-Ramond(R-R) 规范场 $C_{p+1}$ 的荷。$p$ 膜的作用量是质量为 $m$ 、电荷为 $q$ 的粒子在经典场 $A_\mu$ 中作用量的直接推广，形式如下

$$-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^4x-m\int_{\text{trajectory}}ds+q\int_{\text{trajectory}}A, \tag{1}$$

where $A=\int A_\mu dx^\mu=\int \mathbf{A}\cdot\mathbf{v}dt$ and $ds$ is the line element integral. For a $p$ -brane with tension $\tau$ and charge $\mu$ under $C_{p+1}$ , the action (1) is generalized to

其中 $A=\int A_\mu dx^\mu=\int \mathbf{A}\cdot\mathbf{v}dt$ 和 $ds$ 是线元积分。对于一张张力为 $\tau$ 、在 $C_{p+1}$ 下携带电荷 $\mu$ 的 $p$ 膜，作用量 (1) 推广为

$$-\frac{1}{2(2\pi)^7\alpha'^4}\int\sqrt{-g}|F_{p+2}|^2-\tau_p\int_{\text{worldvolume}}\sqrt{-g}+\mu_p\int_{\text{worldvolume}}C_{p+1}, \tag{2}$$

where $F_{p+2}=dC_{p+1}$ . We use mostly plus signature. We also use notations where $\sqrt{-g}|F_{p+2}|^2=F_{p+2}\wedge\star F_{p+2}$.

其中 $F_{p+2}=dC_{p+1}$ 。我们使用 mostly plus 号差，同时采用记号约定 $\sqrt{-g}|F_{p+2}|^2=F_{p+2}\wedge\star F_{p+2}$.

More precisely, the bosonic part of the action for the full theory is given by

更准确地说，完整理论的玻色子部分作用量为

$$\int\mathcal{L}_{\text{bulk}}d^{10}x+\int\mathcal{L}_{\text{DBI}}d^{p+1}x+\int\mathcal{L}_{\text{CS}}d^{p+1}x, \tag{3}$$

where each term can be explained as follows:

其中每一项的解释如下:

- The first term, $\mathcal{L}_{\text{bulk}}$ , is the action of the theory in ten dimensions, namely, type IIA or type IIB supergravity. The terms that are common to both type IIA and IIB are

- 第一项 $\mathcal{L}_{\text{bulk}}$ 是十维理论 (即 IIA 或 IIB 超引力) 的作用量，IIA 和 IIB 共有的项为

$$\frac{1}{(2\pi)^7\alpha'^4}\int d^{10}x\left[e^{-2\phi}\sqrt{-g}\left(R+4(\partial\phi)^2-\frac{1}{2}|H_3|^2\right)-\frac{1}{2}\sqrt{-g}|\widetilde{F}_{p+2}|^2\right]$$

(4)

where $\phi$ is the dilaton, $H_3 = dB_2$ the NS-NS antisymmetric field, and $\widetilde{F}_{p+2}$ is the $p+2$ -form component of $e^{B_2} \wedge \sum_{q=0}^{10} F_q$ . It is also common to use the notation $B_2^{NS} \equiv B_2$ , especially in type IIB where there exists another antisymmetric two-form $B_2^{RR} \equiv C_2$ .

其中 $\phi$ 是 dilaton( dilaton 场，即 dilaton)，$H_3 = dB_2$ 是 NS-NS 反对称场，$\widetilde{F}_{p+2}$ 是 $e^{B_2} \wedge \sum_{q=0}^{10} F_q$ 的 $p+2$ 形式分量。通常也会使用记号 $B_2^{NS} \equiv B_2$ ，这在 IIB 理论中尤为常见，因为 IIB 中存在另一反对称二形式 $B_2^{RR} \equiv C_2$ 。

- The second term is the action for the brane, also known as the Dirac-Born-Infeld (DBI) action. This term contains the coupling to the fields in the NS-NS sector: $g_{\mu\nu}, \phi$ , and $B_{\mu\nu}$ . It also contains the fields that live on the worldvolume of the brane, namely, the gauge field $A_\mu$ with $\mu = 0, \ldots, p$ and the scalar fields $\phi^a$ with $a = p+1, \ldots, 9$ :

- 第二项是膜的作用量，也称为 Dirac-Born-Infeld(DBI) 作用量。该项包含与 NS-NS 区场的耦合: $g_{\mu\nu}, \phi$ 和 $B_{\mu\nu}$ 。它还包含膜世界体上的动力学场，即规范场 $A_\mu$ (满足 $\mu = 0, \ldots, p$ ) 和标量场 $\phi^a$ (满足 $a = p+1, \ldots, 9$ )，形式为:

$$\frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} \int d^{p+1}x e^{-\phi} \sqrt{-\det\left(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} + B_{\mu\nu}\right)}, \tag{5}$$

where $g_{\mu\nu}$ and $B_{\mu\nu}$ are the pullback of the metric and NS-NS field in the bulk, respectively. The fields $\phi^a$ , which parametrize the position of the brane in the transverse space, appear implicitly in $g_{\mu\nu}$ through the pullback.

其中 $g_{\mu\nu}$ 和 $B_{\mu\nu}$ 分别是体空间度规和 NS-NS 场的拉回。参数化膜在横空间中位置的场 $\phi^a$ 通过拉回隐含出现在 $g_{\mu\nu}$ 中。

- The third term, also known as the Chern-Simons (CS) term, contains the couplings to all the fields in the R-R sector [8]. Defining the formal sum

- 第三项也称为 Chern-Simons(CS) 项，包含与所有 R-R 区场的耦合 [8]。定义形式和

$$C = \sum_{k=0}^{9} C_k = C_0 + C_1 + C_2 + \ldots, \tag{6}$$

we can write the brane action $\mathcal{L}_{\text{CS}}$ as

我们可以将膜作用量 $\mathcal{L}_{\text{CS}}$ 写为

$$\frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} \int d^{p+1}x e^{2\pi\alpha' F + B} \wedge C. \tag{7}$$

The first term in the expansion of the exponential corresponds to the charge of the brane under $C_{p+1}$ .

指数展开中的第一项对应膜在 $C_{p+1}$ 下的电荷。

Let us make some comments:

我们做一些说明:

1. By setting $F = B = 0$ and freezing the dilaton to its expectation value in the DBI action, we can read the tension

1. 在 DBI 作用量中令 $F = B = 0$ 并将 dilaton 固定为它的期望值，我们可以得到张力

$$\tau_p = (2\pi)^{-p} \alpha'^{-(p+1)/2} e^{-\phi} \tag{8}$$

of the brane. Note that the brane tension depends on the vacuum value of the dilaton field, $e^{-\phi} = 1/g_s$ , which is the inverse of the string coupling. This shows that D-branes are non-perturbative excitations of string theory, which become very heavy at weak coupling. Notice also the relation between tension and charge of a Dp-brane, which can be read from (7)

膜的。注意膜张力依赖于 dilaton 场的真空值 $e^{-\phi} = 1/g_s$ ，它是弦耦合常数的倒数。这说明 D 膜是弦论的非微扰激发，在弱耦合下会变得非常重。我们还可以从 (7) 中得到 Dp 膜张力与电荷的关系

$$\tau_p = \mu_p / g_s. \tag{9}$$

This relation is a consequence of the BPS condition.

该关系是 BPS 条件的推论。

2. By expanding the DBI action in power series in $\alpha'$ , we can write the effective field theory for $A_\mu$ and $\phi^a$ , which, at the two-derivative level, reduces to the maximally supersymmetric Yang-Mills theory in $p+1$ dimensions. Consider a brane in flat space-time with $B = 0$ lying along the first $p+1$ coordinates. We chose embedding coordinates $X^M = \left\{x^0, \dots, x^p, 2\pi\alpha'\phi_1(x), \dots 2\pi\alpha'\phi_{9-p}(x)\right\}$ , and we let the positions in the transverse space, corresponding to the scalar fields, to fluctuate. We find

2. 将 DBI 作用量按 $\alpha'$ 做幂级数展开后，我们可以写出 $A_\mu$ 和 $\phi^a$ 的有效场论，该理论在二导数阶下约化为 $p+1$ 维的最大超对称杨-米尔斯理论。考虑平坦时空内的一张膜，其中 $B = 0$ 沿前 $p+1$ 个坐标延伸。我们选取嵌入坐标 $X^M = \left\{x^0, \dots, x^p, 2\pi\alpha'\phi_1(x), \dots 2\pi\alpha'\phi_{9-p}(x)\right\}$，并让对应标量场的横向空间位置发生涨落，可得

$$g_{\mu\upsilon} = g_{MN} \partial_\mu X^M \partial_\upsilon X^N = \eta_{\mu\upsilon} + (2\pi\alpha')^2 \partial_\mu \phi^a \partial_\upsilon \phi^a \tag{10}$$

and the DBI action at the two-derivative order becomes

二导数阶下的 DBI 作用量为

$$-\frac{1}{2g_{\rm YM}^2} \int d^{p+1}x \left(F_{\mu\upsilon}F^{\mu\upsilon} + 2\partial\phi^a\partial\phi^a\right), \tag{11}$$

where the gauge coupling

其中规范耦合

$$g_{\mathrm{YM}}^{2}=2(2\pi)^{p-2}\alpha'^{(p-3)/2}g_s \tag{12}$$

is proportional to the string coupling $g_s$. In general, a system of $N$ branes describes the Yang-Mills theory with $\mathrm{U}(N)$ gauge group, and a trace over color indices should be added in the previous formula.

正比于弦耦合常数 $g_s$。一般来说，$N$ 张膜的系统对应规范群为 $\mathrm{U}(N)$ 的杨-米尔斯理论，且上述公式需要额外添加色指标的迹。

3. In addition to the dilaton, also, the NS-NS and R-R antisymmetric bulk fields appear as external coupling and parameters. Schematically, for a brane in flat space-time, the gauge field couples to the bulk fields at the two-derivative level as

3. 除 dilaton 外，NS-NS 和 R-R 反对称体场也作为外耦合和参数出现在作用量中。形式上，对于平坦时空内的膜，规范场在二导数阶下与体场的耦合为

$$\int d^{p+1}x\left[e^{-\phi}\left((2\pi\alpha'F+B)\wedge*(2\pi\alpha'F+B)\right)+C\wedge e^{2\pi\alpha'F+B}\right]. \tag{13}$$

Only the $(p+1)$ -form part of $C\wedge e^{2\pi\alpha'F+B}$ contributes to the action; for example, for a D3-brane, we have

只有 $C\wedge e^{2\pi\alpha'F+B}$ 的 $(p+1)$ 形式部分对作用量有贡献；例如，对于 D3 膜，我们得到

$$C_4+C_2\wedge(2\pi\alpha'F+B)+\frac{C_0}{2}(2\pi\alpha'F+B)\wedge(2\pi\alpha'F+B). \tag{14}$$

4. The generalization of the DBI and CS action to a system of $N$ branes is subtle. At the level of the two-derivative effective action, stacking $N$ branes on top of each other simply amounts to replacing an abelian U(1) theory with a non-abelian $\mathrm{U}(N)$ theory. However, the full DBI action can be written only in the abelian case; there is no known complete generalization to the non-abelian case. Notice also that the $2\pi\alpha'F+B$ term can be written only for an abelian group and $B$ is only coupled to the $\mathrm{U}(1)$ factor, corresponding to the center of mass of the system.

4. 将 DBI 作用量和 CS 作用量推广到 $N$ 张膜的系统十分复杂。在二导数有效作用量层面，将 $N$ 张膜堆叠在一起，仅相当于把阿贝尔 U(1) 理论替换为非阿贝尔 $\mathrm{U}(N)$ 理论。但完整的 DBI 作用量仅能在阿贝尔情形下写出；目前尚不存在非阿贝尔情形下已知的完整推广。还需注意，$2\pi\alpha'F+B$ 项仅能在阿贝尔群下写出，且 $B$ 仅耦合到对应系统质心的 $\mathrm{U}(1)$ 因子。

5. The brane can be regarded as a source of the fields in the bulk. In analogy to the electrostatic case in four dimensions, where a point charge is a source of the gauge field, $d*F=\delta^{(3)}(x)$ and $F\sim q/r^2$, the brane is a sources for $g_{\mu\nu}$ and $C_{p+1}$. The equations of motion schematically read

5. 膜可以看作体场的源。类似于四维静电学中点电荷是规范场 $d*F=\delta^{(3)}(x)$ 和 $F\sim q/r^2$ 的源，膜是 $g_{\mu\nu}$ 和 $C_{p+1}$ 的源。运动方程形式上可以写为

$$\frac{1}{2}\int d^{10}x\sqrt{-g}|F_{p+2}|^2 + q\int d^{p+1}xC_{p+1} \rightarrow d * F_{p+2} = q\delta^{(9-p)}(x) \tag{x}$$

$$\int d^{10}x\sqrt{-g}R + \tau\int d^{p+1}x\sqrt{-g} \ \rightarrow \ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \propto \tau\delta$$

(15)

We also have a similar relation for the dilaton

dilaton 也满足类似的关系

$$\Box\phi \sim R \sim \text{ a delta function} \tag{16}$$

Therefore, $g_{\mu\nu}, F_{p+2}$ , and $\phi$ behave like inverse powers of $r$ , the distance to the brane in the transverse $(9-p)$ directions.

因此 $g_{\mu\nu}, F_{p+2}$ 和 $\phi$ 表现为 $r$ 的负幂次，其中 $r$ 是横向 $(9-p)$ 方向上到膜的距离。

## Explanation of $2\pi\alpha^{'}F + B$

## $2\pi\alpha^{'}F + B$ 的阐释

It is a general phenomenon that an extended charged object cannot have a finite size. The uniformly distributed charge carried by such an object cannot terminate on the end point, but it has to be absorbed by something. This is a direct consequence of the Gauss law. The equation of motion of the $p$ -brane reads

延伸带电物体无法拥有有限尺寸是普遍现象。这类物体携带的均匀分布电荷无法终止于端点，必须被某种物体吸收，这是高斯定律的直接推论。$p$ 膜的运动方程为

$$d * F_{p+2} = q\delta^{(9-p)}(\mathbf{x}) \tag{17}$$

where $\mathbf{x}$ is the position of the brane. The Gauss law follows from the integration over the transverse space $\mathbb{R}^{9-p}$ to the brane:

其中 $\mathbf{x}$ 是膜的位置。高斯定律可由对膜的横向空间 $\mathbb{R}^{9-p}$ 积分得到:

$$q = \int_{\mathbb{R}^{9-p}} d * F_{p+2} = \int_{S^{8-p}} * F_{p+2} \tag{18}$$

where this integral is evaluated at a point $z$ on the worldvolume $\mathbb{R}^{p+1}$ of the brane and is independent of $z$ . If the brane had an end, we could move the sphere after the end point until the integral becomes zero, and we would obtain a contradiction. Another way to see this is as follows: Suppose that the brane is semi-infinite and ends at the coordinate $y = 0$ . The equation of motion is modified to

该积分在膜世界体 $\mathbb{R}^{p+1}$ 上的点 $z$ 处计算，结果与 $z$ 无关。如果膜存在端点，我们可以将球面移过端点直至积分变为零，从而得到矛盾。另一种理解方式如下: 假设膜是半无限的，终止于坐标 $y = 0$ 处，运动方程会被修改为

$$d * F_{p+2} = q\delta^{(9-p)}(\mathbf{x})\,\theta(y)\,. \tag{19}$$

Differentiating, we obtain the contradiction

求导后我们得到矛盾

$$0 = q\delta^{(10-p)}(\mathbf{x}, y)\,. \tag{20}$$

Consider now an open string ending on a D-brane. The fundamental string is charged under the field $B_{\mu\nu}$ in the NS-NS sector. Since it has an end point, this may seem at first sight to be inconsistent. Neglecting coefficients, one would write the equation of motion as

现在考虑终止于 D 膜上的开弦。基本弦在 NS-NS sector 下对场 $B_{\mu\nu}$ 荷电。由于弦存在端点，乍看这似乎不自洽。忽略系数的话，运动方程可以写为

$$d * H = \delta^{(8)}(\mathbf{x})\,\theta(x_1) \tag{21}$$

where $x_1$ is the spatial coordinate along the string. However, from (13), we see that there should in fact be another term coming from the equation of motion of $B_{\mu\nu}$ :

其中 $x_1$ 是沿弦方向的空间坐标。但从 (13) 可以看出，实际上应该存在另一项，来自 $B_{\mu\nu}$ 的运动方程:

$$d * H = \delta^{(8)}(x)\,\theta(x_1) + \delta^{(9-p)}(\mathbf{x}_0) * (2\pi\alpha' F + B)\,, \tag{22}$$

where $\mathbf{x}_0$ is the position of the D-brane. Differentiating, we obtain

其中 $\mathbf{x}_0$ 是 D 膜的位置。求导后得到

$$d * (2\pi\alpha' F + B) = \text{ some delta function.} \tag{23}$$

Therefore, everything is consistent if we turn on the field $F$ on the brane in such a way that it satisfies this equation, namely, the end point of the open string appears to the field $F$ as a charged particle. In conclusion, the coupling $2\pi\alpha' F + B$ is needed for consistency for an open string to end on a D-brane.

因此，如果我们在膜上开启场 $F$ ，使其满足该方程——即开弦端点在场 $F$ 看来就是一个带电粒子——那么一切就都是自洽的。综上，为了让开弦能终止于 D 膜上同时保证自洽，耦合 $2\pi\alpha' F + B$ 是必需的。

## Branes Ending on Branes

## 终止于膜的膜

Strominger [9] has discovered that the same argument allows certain $p$ -brane to end on another brane whose dimension is larger. The charge of the former brane is absorbed by turning on the field on the larger brane in an appropriate manner. All such configurations arise by applying a chain of $T$ and $S$ dualities to the basic example of a fundamental string ending on a Dp-brane.

斯特罗明格 [9] 发现，相同的论证表明特定的 $p$ 膜可以终止于另一张维度更高的膜上。前者的电荷可以通过在大膜上以合适的方式开启场来吸收。所有这类构型都可以通过对基本弦终止于 Dp 膜的基础例子应用一系列 $T$ 和 $S$ 对偶得到。

As an example, let us consider a D1-brane ending on a D3-brane in type IIB. This configuration can be obtained by applying $S$ -duality to a fundamental string ending on a D3-brane. The D1-brane is charged under the field $C_2$ in the R-R sector. The equation of motion is of the form

举一个例子，我们来考虑 IIB 型理论中终止于 D3 膜的 D1 膜。这个构型可以通过对终止于 D3 膜的基本弦应用 $S$ 对偶得到。D1 膜带有 R-R sector 中场 $C_2$ 的电荷。运动方程形如

$$d * F_3 = \delta^{(8)}(\hat{x}, y)\,\theta\left(x^1\right), \tag{24}$$

where $x^1$ is the spatial coordinate on the D1-brane, $y$ are the three coordinates that give the position of the D1-brane inside the D3-brane worldvolume, and $\hat{x}$ are the coordinates in the transverse directions to the D3-brane excluding $x^1$ . Again, the above equation is not consistent. The modification along the line of (13) gives

其中 $x^1$ 是 D1 膜上的空间坐标，$y$ 是确定 D1 膜在 D3 膜世界体积内位置的三个坐标，$\hat{x}$ 是 D3 膜横向上除去 $x^1$ 的坐标。上述方程仍然不自洽。按照 (13) 的思路修改后得到

$$d * F_3 = \delta^{(8)}(\hat{x}, y)\,\theta\left(x^1\right) + \left(2\pi\alpha' F + B\right)\delta^{(6)}\left(\hat{x}, x^1\right), \tag{25}$$

since the D3-brane sits at $\hat{x} = x^1 = 0$ . Differentiating this equation, we have

因为 D3 膜位于 $\hat{x} = x^1 = 0$ 处。对该方程求导可得

$$d\left(2\pi\alpha' F + B\right) = \delta^{(3)}(y). \tag{26}$$

Thus, we see that the D1-brane can be regarded as a particle (monopole) magnetically charged under the field $F$ of the D3-brane.

因此我们可以看到，D1 膜可以被视作 D3-brane 的场 $F$ 下带磁荷的粒子 (磁单极)。

Similarly, one can show, in general, that a D$p$ -brane can end on a D$(p+2)$ -brane. Using $S$ -duality, a system of a D3-brane ending on a D5-brane can be mapped to a system of a D3-brane ending on an NS5-brane.

Applying $T$ -duality along any direction of NS5-brane worldvolume, we see that this system is mapped to a system of a D$p$ -brane (with $p \leq 6$ ) ending on the NS5-brane.

同理，一般情况下可以证明，D$p$ 膜可以终止于 D$(p+2)$ 膜上。利用 $S$ 对偶，终止于 D5 膜的 D3 膜系统可以映射为终止于 NS5 膜的 D3 膜系统。沿 NS5 膜世界体积的任意方向应用 $T$ 对偶，我们可以看到该系统被映射为 (带有 $p \leq 6$ 的) D$p$ 膜终止于 NS5 膜的系统。

In general, when a D-brane ends on another D-brane, the smaller brane can be regarded as a monopole from the perspective of the larger brane. However, when an open string ends on a D-brane, the former appears as an electrically charged object from the perspective of the latter.

一般而言，当 D 膜终止于另一张 D 膜上时，从小膜的角度看，它可以被视作大膜视角下的磁单极。而当开弦终止于 D 膜上时，从 D 膜的视角看，开弦是一个带电荷的物体。

## A Note About Conventions

## 关于约定的说明

Not to clutter notations and to make the physical interpretation as clear as possible, in the rest of this lecture, we will set $\alpha'$ to an appropriate value, and we will be cavalier about signs and coefficients.

为了避免符号体系过于繁杂，同时尽可能清晰地呈现物理解释，在本讲义剩余内容中，我们会将 $\alpha'$ 设为合适的值，并且对符号和系数不作严格苛求。

## Realization of Gauge Theories

## 规范理论的实现

The field theory living in the worldvolume of a stack of $N$D$p$ -brane in type IIA or IIB string theory is a $(p+1)$ -dimensional gauge theory with 16 supercharges and group $\mathrm{U}(N)$ . This corresponds to $\mathcal{N}=4$ in four dimensions or $\mathcal{N}=8$ in three dimensions.

IIA 或 IIB 弦论中一堆 $N$D$p$-膜的世界体积上存在的场论是一个具有 16 个超荷、群为 $\mathrm{U}(N)$ 的 $(p+1)$ 维规范理论。这对应四维的 $\mathcal{N}=4$ 或三维的 $\mathcal{N}=8$ 。

The amount of supersymmetry can be understood as follows: The theory in the ten-dimensional bulk dimensionally reduced to $(p+1)$ dimensions would have 32 supercharges; however, a D-brane, which is a BPS object, breaks half of the supersymmetry. To be precise, it preserves the supercharges that satisfy

超对称数量可以这样理解: 十维 bulk 中的理论维约化到 $(p+1)$ 维原本会有 32 个超荷；但 D-膜作为 BPS 对象会破缺一半超对称。准确来说，它保留满足下式的超荷:

$$\varepsilon_L = \Gamma_0 \Gamma_1 \cdots \Gamma_p \varepsilon_R \tag{27}$$

where $\varepsilon_{L,R}$ are the two spinors of the theory in ten dimensions. Note that they have the same chirality in type IIB and opposite chirality in type IIA.

其中 $\varepsilon_{L,R}$ 是十维理论的两个旋量。注意: 它们在 IIB 型中手征性相同，在 IIA 型中手征性相反。

From the point of view of a worldvolume observer, the fields of the theory in the bulk that are coupled to the D-brane become background fields, and they are non-dynamical. A way to understand this is to regularize the theory by putting it in a finite volume $V^{9-p}$ . In this case, from the brane point of view, the kinetic term of the fields in the bulk is multiplied by $V^{9-p}$ , and so the coupling tends to zero as $V$ goes to infinity. In other words, they are frozen. Although these fields are nondynamical, they have an effect on the brane. Apart from the values of $g_{ij}, B_{ij}$ , etc., which describe the embedding in space-time, the object that mostly affects the brane is the vacuum expectation value (VEV) of the various scalar fields in the bulk. In general, this becomes a parameter in the gauge theory. For example, the VEV of the dilaton, whose coupling is via the term $\int \sqrt{-g} e^{-\phi} F_{\mu\nu}^2 d^{p+1}x$ , gives the coupling constant of the gauge theory, $\frac{1}{g^2} \propto e^{-\phi} = \frac{1}{g_s}$ , where $g_s$ is the string coupling.

从世界体积观测者的角度看，bulk 中与 D-膜耦合的理论场会成为背景场，且它们是非动力学的。理解这一点的一种方法是通过将理论置于有限体积 $V^{9-p}$ 来正则化。这种情况下，从膜的角度看，bulk 场的动能项会乘上 $V^{9-p}$ ，因此当 $V$ 趋于无穷时耦合趋于零，换句话说这些场被冻结了。尽管这些场是非动力学的，它们仍会对膜产生影响。除了描述膜在时空中嵌入的 $g_{ij}, B_{ij}$ 等量之外，对膜影响最大的是 bulk 中各类标量场的真空期望值 (VEV)。一般来说，这个真空期望值会成为规范理论中的一个参数。例如，通过项 $\int \sqrt{-g} e^{-\phi} F_{\mu\nu}^2 d^{p+1}x$ 耦合的 dilation 的真空期望值给出了规范理论的耦合常数 $\frac{1}{g^2} \propto e^{-\phi} = \frac{1}{g_s}$ ，其中 $g_s$ 是弦耦合。

In order to obtain a theory with eight supercharges, we need to reduce the amount of supersymmetry by a half. This can be done in several ways. Let us discuss here the method of adding a different type of branes.

若要得到具有 8 个超荷的理论，我们需要将超对称数量再减少一半，这可以通过多种方式实现。我们在这里讨论添加不同类型膜的方法。

We consider a system of a D$p$ -brane and a D$p'$ -brane, such that $p < p'$, $p$ , and $p'$ are both even or both odd (so that we are in type IIA or IIB, respectively) and both branes have $p$ directions in common. We can determine the amount of supersymmetry that this system preserves as follows:

我们考虑一个由 D$p$ -膜和 D$p'$ -膜组成的系统，满足 $p < p'$, $p$ ，且 $p'$ 同偶或同奇 (因此我们分别处于 IIA 型或 IIB 型弦论中)，两个膜共有 $p$ 个公共方向。我们可以按如下方式确定这个系统保留的超对称数量:

$$\varepsilon_L = \Gamma_0 \cdots \Gamma_p \varepsilon_R \tag{28}$$

$$\varepsilon_L = \Gamma_0 \cdots \Gamma_p \Gamma_{p+1} \cdots \Gamma_{p'} \varepsilon_R$$

This implies that

这意味着

$$\varepsilon_R = \Gamma_{p+1}\cdots\Gamma_{p'}\varepsilon_R \equiv P\varepsilon_R, \tag{29}$$

where we define $P = \Gamma_{p+1}\cdots\Gamma_{p'}$ .

其中我们定义 $P = \Gamma_{p+1}\cdots\Gamma_{p'}$ 。

We see that $P^2 = -1$ if $p' - p = 2, 6, 10, \cdots$ , in which case we have $\varepsilon_R = P^2\varepsilon_R = -\varepsilon_R$ , meaning that supersymmetry is completely broken.

我们看到: 如果 $p' - p = 2, 6, 10, \cdots$ ，则 $P^2 = -1$ ，此时我们得到 $\varepsilon_R = P^2\varepsilon_R = -\varepsilon_R$ ，意味着超对称被完全破缺。

On the other hand, if $p' = p (\mod 4)$ , then $P^2 = 1$ . Since $P$ is also traceless, we conclude that half of its 16 eigenvalues are +1 and the other half are -1 . The relation (29) implies that $P$ preserves 8 (out of 16) components of $\varepsilon_R$ . Moreover, given $\varepsilon_R$ , the first relation of (28) fixes completely $\varepsilon_L$ . Thus, this system of branes preserves eight supercharges in total.

另一方面，如果 $p' = p (\mod 4)$ ，则 $P^2 = 1$ 。由于 $P$ 也是无迹的，我们可以推导出它的 16 个本征值中有一半是 +1，另一半是-1。式 (29) 表明 $P$ 保留了 $\varepsilon_R$ 的 8 个 (共 16 个) 分量。此外，给定 $\varepsilon_R$ ，式 (28) 的第一个关系式完全确定了 $\varepsilon_L$ 。因此这个膜系统总共保留了 8 个超荷。

This observation, in fact, follows from the following general statement: If the number of coordinates corresponding to the Dirichlet-Neumann (DN) or Neumann-Dirichlet (ND) boundary conditions of an open string is a multiple of 4, then the corresponding brane system preserves one quarter of the supersymmetry.

实际上，这一观测结果可由下述一般性结论推出: 若开弦满足狄利克雷-诺依曼 (DN) 或诺依曼-狄利克雷 (ND) 边界条件对应的坐标数为 4 的倍数，则相应的膜系会保留四分之一的超对称性。

Other methods to obtain gauge theories with eight supercharges involve considering D $p$ -branes at orbifold singularities or D $p$ -branes stretched between two parallel NS5-branes. We will discuss some of these techniques in sections " $\mathcal{N} = 2$ Gauge

其他得到带有 8 个超荷的规范理论的方法包括考虑位于轨形奇点处的 D $p$ -膜，或是张在两张平行 NS5-膜之间的 D $p$ -膜。我们会在“$\mathcal{N} = 2$ 规范

Theories in Four Dimensions", "Hanany-Witten Brane Configurations in Three Dimensions", and "Hanany-Witten Brane Configurations in Four Dimensions".

四维理论”、“三维中的 Hanany-Witten 膜构型”和“四维中的 Hanany-Witten 膜构型”章节中讨论其中部分方法。

In these notes, we will focus on the bosonic content of the worldvolume theories. The fermionic content can be easily reconstructed from supersymmetry, and it will be mentioned only when needed.

在这些讲义中，我们将聚焦于世界体积理论的玻色子内容。费米子内容可很容易地通过超对称性重构，仅在必要时才会提及。

## $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory in Four Dimensions

## $\mathcal{N} =$ 四维 N=4 超对称杨-米尔斯理论

An $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) with gauge group $U(N)$ is realized on $N$ parallel D3-branes in type IIB string theory. The bosonic fields of the theory consist in a gauge field $A_\mu$ and six real scalars $\phi^a$ (with $a = 1, \ldots, 6$ ) that parametrize the positions of the $N$ branes. They all transform in the adjoint representation of the gauge group. The moduli space of supersymmetric vacua consists of a Coulomb branch in which the VEVs $\langle\phi^a\rangle \neq 0$ and the $\mathrm{U}(N)$ gauge group is broken to $\mathrm{U}(1)^N$ . One can perform a gauge transformation to put $\langle\phi^a\rangle$ in the Cartan subalgebra

在 IIB 型弦论中，$\mathcal{N} = 4$ 张平行 D3 膜上实现了规范群为 $U(N)$ 的超对称杨-米尔斯理论 (SYM)。该理论的玻色场包含一个规范场 $A_\mu$ 与六个实标量场 $\phi^a$ (满足 $a = 1, \ldots, 6$ )，这些标量场参数化了 $N$ 张膜的位置。它们全部属于规范群的伴随表示。超对称真空的模空间包含库仑分支，在此分支中真空期望值 $\langle\phi^a\rangle \neq 0$ 使原 $\mathrm{U}(N)$ 规范群破缺为 $\mathrm{U}(1)^N$ 。我们可以通过规范变换将 $\langle\phi^a\rangle$ 置于嘉当子代数中

$$\langle\phi^a\rangle = \begin{pmatrix} \phi_1^a & & \\ & \ddots & \\ & & \phi_N^a \end{pmatrix}, \tag{30}$$

where the values of $\phi_i^a$ , with $i = 1, \ldots, N$ , correspond to the positions of the brane. When the $\phi_a$ are all different, the branes are in different positions, and the gauge group is $\mathrm{U}(1)^N$ . When all $\phi_i^a = 0$ , the branes are coincident, and the gauge group becomes $\mathrm{U}(N)$ .

其中 $\phi_i^a$ 的取值对应于膜的位置，满足 $i = 1, \ldots, N$ 。当所有 $\phi_a$ 各不相同时，膜分散在不同位置，规范群为 $\mathrm{U}(1)^N$ 。当所有 $\phi_i^a = 0$ 相等时，膜重合，规范群变为 $\mathrm{U}(N)$ 。

## Realization of the S-Duality in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory

## $\mathcal{N} = 4$ 超对称杨-米尔斯理论中 S 对偶的实现

The type IIB string theory in ten dimensions has an $\mathrm{SL}(2,\mathbb{Z})$ symmetry that rotates $(B_2, C_2)$ as a doublet as well as transforms the complex parameter (also known as the axio-dilaton field)

十维 IIB 型弦理论具有 $\mathrm{SL}(2,\mathbb{Z})$ 对称性，该对称性将 $(B_2, C_2)$ 旋转为二重态，并对复参数 (也称为轴子- dilaton 场) 进行变换

$$\tau = ie^{-\phi} + C_0, \tag{31}$$

in the following way:

变换方式如下:

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B \\ C_2 \end{pmatrix}, \ \tau \to \frac{a\tau + b}{c\tau + d}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{32}$$

The fundamental string, which is charged under $B_2$ , and the D1-brane, which is charged under $C_2$ , get exchanged among each other. Similarly, the NS5-brane, which is magnetically charged under $B_2$ , and the D5-brane, which is magnetically charged under $C_2$ , also get exchanged. On the other hand, the D3-brane, which is charged under $C_4$ , is invariant under this action.

携带 $B_2$ 荷的基本弦与携带 $C_2$ 荷的 D1 膜相互交换。类似地，携带 $B_2$ 磁荷的 NS5 膜与携带 $C_2$ 磁荷的 D5 膜也相互交换。另一方面，携带 $C_4$ 荷的 D3 膜在该变换下保持不变。

For the theory on the $N$ D3-branes, the axio-dilaton field $\tau$ in (31) is identified with the complexified gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi}{g_{\mathrm{YM}}^2} i$ of the SYM theory. Indeed, the dilaton field $e^{-\phi}$ corresponds to the gauge coupling $\frac{4\pi}{g_{\mathrm{YM}}^2}$ of the theory according to (12), and the R-R scalar $C_0$ corresponds to the $\theta$ -angle of the theory via the term

对于 $N$ 张 D3 膜上的理论，(31) 式中的轴子-dilaton 场 $\tau$ 等同于 SYM 理论的复化规范耦合 $\tau = \frac{\theta}{2\pi} + \frac{4\pi}{g_{\mathrm{YM}}^2} i$。事实上，根据 (12) 式，dilaton 场 $e^{-\phi}$ 对应理论的规范耦合 $\frac{4\pi}{g_{\mathrm{YM}}^2}$，而 R-R 标量 $C_0$ 通过作用量 (13) 中的项对应理论的 $\theta$ 角

$$\frac{1}{4\pi} \int C_0 \, \mathrm{Tr} \, F \wedge F = \frac{\theta}{8\pi^2} \int \mathrm{Tr} \, F \wedge F \tag{33}$$

in the action (13). In particular, under the action of the $S$ generator of $\mathrm{SL}(2, \mathbb{Z})$ , the theory on the $N$ D3-brane transforms into itself, while the coupling is transformed as $\tau \to -1/\tau$ . Explicitly, the matrix representations of the $S$ and $T$ generators of $\mathrm{SL}(2, \mathbb{Z})$ can be taken as $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Strictly speaking, the theory with gauge group $\mathrm{SU}(N)$ is mapped by the $S$ generator to the theory with gauge group $\mathrm{SU}(N)/\mathbb{Z}_N$ . From holographic perspective, it was pointed out in [10] that the $\mathcal{N} = 4$ SYM theories with different gauge groups that correspond to the same gauge algebra $\mathfrak{su}(N)$ can be realized by different choices of boundary conditions for the NS-NS and R-R 2-form gauge fields $B_2$ and $C_2$ in $\mathrm{AdS}_5$ . In fact, any pair of such theories that is related by an $\mathrm{SL}(2, \mathbb{Z})$ transformation corresponds to the boundary conditions on $(B_2, C_2)$ that is related by the same $\mathrm{SL}(2, \mathbb{Z})$ element in type IIB string theory. We refer the reader to [11, Section 2.4] for a more complete discussion regarding the $\mathrm{SL}(2, \mathbb{Z})$ action on the $\mathcal{N} = 4$SYM theory as well as the precise form of the gauge group.

在作用量 (13) 中。特别地，在 $\mathrm{SL}(2,\mathbb{Z})$ 的 $S$ 生成元作用下，$N$ 张 D3 膜上的理论变换为自身，而耦合变换为 $\tau \to -1/\tau$ 。具体来说，$\mathrm{SL}(2,\mathbb{Z})$ 的 $S$ 生成元和 $T$ 生成元的矩阵表示可取为 $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ 和 $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$。严格来说，规范群为 $\mathrm{SU}(N)$ 的理论被 $S$ 生成元映射到规范群为 $\mathrm{SU}(N)/\mathbb{Z}_N$ 的理论。从全息角度，文献 [10] 指出，对应同一规范代数 $\mathfrak{su}(N)$ 、不同规范群的 $\mathcal{N}=4$ SYM 理论，可以通过对 $\mathrm{AdS}_5$ 中 NS-NS 和 R-R 2-形式规范场 $B_2$ 和 $C_2$ 选择不同边界条件来实现。实际上，任意一对通过 $\mathrm{SL}(2,\mathbb{Z})$ 变换关联的此类理论，都对应 IIB 弦理论中通过同一 $\mathrm{SL}(2,\mathbb{Z})$ 元素关联的 $(B_2, C_2)$ 上的边界条件。关于 $\mathrm{SL}(2,\mathbb{Z})$ 对 $\mathcal{N}=4$SYM 理论的作用以及规范群的精确形式，读者可以参考 [11, 第 2.4 节] 获得更完整的讨论。

It is also easy to see that, under the $S$ action, electrically charged excitations are mapped to magnetically charged excitations. The former correspond to the $W$ -bosons which can be realized as fundamental strings stretched between two separated parallel D3-branes. Indeed, as we saw, the end of the fundamental string can be viewed as an electrically charged particle from the perspective of a D3-brane observer. Applying the $S$ generator to this system, we obtain a system with D1- branes stretched between two separated D3-branes. The end of the D1-brane appears as a monopole, which is a magnetically charged particle, from the perspective of a D3-brane observer. In terms of worldvolume fields, the action of $S$ -duality is $F_{\mu\nu} \leftrightarrow \varepsilon_{\mu\nu\tau\rho}F^{\tau\rho}$.

不难发现，在 $S$ 作用下，带电激发会被映射为带磁激发。前者对应 $W$ 玻色子，可实现为张开在两个分离平行 D3 膜之间的基本弦。正如我们所见，从 D3 膜观测者的角度来看，基本弦的端点可以被视为一个带电粒子。对该系统作用 $S$ 生成元后，我们得到了一个 D1 膜张开在两个分离 D3 膜之间的系统。从 D3 膜观测者的角度来看，D1 膜的端点表现为磁单极，也就是一个带磁粒子。对世界体场而言，$S$ 对偶的作用是 $F_{\mu\nu} \leftrightarrow \varepsilon_{\mu\nu\tau\rho}F^{\tau\rho}$.

## $\mathcal{N}=2$ Gauge Theories in Four Dimensions

## $\mathcal{N}=$ 二维四维规范理论

A 4 d$\mathcal{N}=2$ gauge theory has eight real supercharges, and it can be realized by dimensionally reducing a 6 d$\mathcal{N}=(1,0)$ theory to four dimensions. The 6d theory has a SU(2) R-symmetry and two types of multiplets, the vector multiplet with just a gauge field and the hypermultiplet, containing four real scalars. (As usual, we focus on the bosonic fields.)

一个 4 d$\mathcal{N}=2$ 规范理论有八个实超荷，它可以通过将 6 d$\mathcal{N}=(1,0)$ 理论维数约化到四维实现。该六维理论具有 SU(2) R 对称性，包含两类多重态: 仅含规范场的矢量多重态，以及包含四个实标量的超多重态。(和通常情况一样，我们聚焦于玻色场。)

The 4d theory has $R$ -symmetry $\mathrm{U}(1) \times \mathrm{SU}(2)$ , where $\mathrm{SU}(2)$ already exists in 6 d and $\mathrm{U}(1)$ corresponds to the rotational symmetry in the two directions used in going to four dimensions. There are two types of supermultiplets obtained by reducing the 6d ones. The first one is the hypermultiplet, containing four real scalars transforming only under the SU(2) symmetry as in 6d. The second one is the vector multiplet. By reducing the 6 d gauge field, we obtain a 4 d gauge field $A_\mu$ and a complex scalar $\varphi$ , charged under $\mathrm{U}(1)$ . The moduli space of supersymmetric vacua consists of a Coulomb branch, where $\varphi$ has a VEV and the gauge group is generically broken to the Cartan subgroup, and a Higgs branch, parametrized by the VEVs of the hypermultiplet scalar fields. Supersymmetry requires that the Higgs branch is an hyperKähler manifold. The Higgs branch is not renormalized by quantum effects [12, 13], while the Coulomb branch is. There can exist mixed branches.

该四维理论具有 $R$ 对称性 $\mathrm{U}(1)\times\mathrm{SU}(2)$，其中 SU(2) 已存在于 6 d 中，U(1) 对应降维到四维过程中所用到的两个额外方向的旋转对称性。约化六维多重态可得到两类超多重态: 第一类是超多重态，包含四个实标量，和六维情况一样仅在 SU(2) 对称性下变换。第二类是矢量多重态，通过约化 6 d 规范场，我们得到一个 4 d 规范场 $A_\mu$ 和一个在 U(1) 下带荷的复标量 $\varphi$。超对称真空的模空间由库仑分支和希格斯分支构成: 库仑分支上 $\varphi$ 获得真空期望值，规范群一般破缺到嘉当子群；希格斯分支由超多重态标量场的真空期望值参数化。超对称性要求希格斯分支是超凯勒流形。和库仑分支不同，希格斯分支不会被量子效应重整化 [12, 13]。也可以存在混合分支。

The hypermultiplets have the same description in three, four, five, and six spacetime dimensions. Not being corrected quantum mechanically, the Higgs branch for theories with eight supercharges is the same in all dimensions. (Note: the Higgs branch can be enhanced at particular points of the full moduli space if new degrees of freedom become massless. An example of this phenomenon in (3+1)-dimensions occurs at a point where the mutually local states become massless [14]. This appears, for instance, at the maximal superconformal point, also known as the Argyres-Douglas point [15], of the 4 d$\mathcal{N}=2$ pure $\mathrm{SU}(2n)$ super Yang-Mills with $n\geq 2$ [16].) In particular, the system with D$p-+$D$(p+4)$ -branes has the same Higgs branch for all $p\leq 6$ . In a theory with four supercharges, the supersymmetric vacua are obtained by setting to zero the F-terms and the D-terms and modding by the gauge group $G$ . In a theory with eight supercharges, the (complex) F-terms and the D-terms combine into a triplet of D-terms $\mathbf{D}$ , transforming under SU(2), and the Higgs branch is then given by

超多重态在三、四、五、六时空维度中具有相同的描述。由于不被量子力学修正，拥有八个超荷的理论的希格斯分支在所有维度中都是相同的。(注: 如果新自由度变为无质量，希格斯分支可以在全模空间的特殊点增强。该现象的一个例子出现在 (3+1) 维中互局域态变为无质量的点 [14]，例如，该现象出现在带有 $n\geq 2$ 的 4 d$\mathcal{N}=2$ 纯 $\mathrm{SU}(2n)$ 超杨-米尔斯理论的最大超共形式点，也即阿吉里斯-道格拉斯点 [15, 16]。) 特别地，带有 D$p-+$D$(p+4)$ 膜的系统对所有 $p\leq 6$ 都具有相同的希格斯分支。在带有四个超荷的理论中，超对称真空通过将 F 项和 D 项置零，再模去规范群 $G$ 得到。在带有八个超荷的理论中，(复)F 项和 D 项组合为在 SU(2) 下变换的 D 项三元组 $\mathbf{D}$，此时希格斯分支由下式给出

$$\text{Higgs branch } = \{\mathbf{D}=0\}/G\,. \tag{34}$$

This construction is known as hyperKähler quotient [17], and it is an efficient method to produce hyperKähler manifolds. In mathematical language, the D-terms $\mathbf{D}$ are (hyperKähler) moment maps for the action of the gauge group $G$ on the space of free hypermultiplets.

该构造被称为超凯勒商 [17]，是构造超凯勒流形的有效方法。用数学语言来说，D 项 $\mathbf{D}$ 是规范群 $G$ 作用在自由超多重态空间上的 (超凯勒) 矩映射。

For future reference, we notice that one can also introduce Fayet-Iliopoulos (FI) parameters. In theories with four supercharges, the FI parameters appear in the Lagrangian through a linear coupling to the D-term, namely, $\int \zeta D$ . They exist only if the gauge group contains U(1) factors. In theories with eight supercharges, the D-terms form a triplet $\mathbf{D}$ , and so $\boldsymbol{\zeta}$ is also a triplet, and both transform under SU(2). The equations that determine the VEVs of the hypermultiplets in a supersymmetric vacuum are now generalized to

为方便后续参考，我们注意到还可以引入费耶特-伊利亚 opoulos(Fayet-Iliopoulos，FI) 参数。在含有 4 个超荷的理论中，FI 参数通过与 D 项的线性耦合出现在拉格朗日量中，即 $\int \zeta D$ 。仅当规范群包含 U(1) 因子时，它们才存在。在含有 8 个超荷的理论中，D 项构成一个三重态 $\mathbf{D}$ ，因此 $\boldsymbol{\zeta}$ 也是一个三重态，二者都在 SU(2) 下变换。现在确定超对称真空中超多重标量真空期望值的方程推广为

$$\mathbf{D}(x) - \boldsymbol{\zeta} = 0. \tag{35}$$

The FI parameters generically lift the Coulomb branch and smoothen the singularities at the origin of the Higgs branch.

FI 参数通常会抬升库仑分支，并平滑化希格斯分支原点处的奇点。

## Example: A D3-D7 System

## 示例:D3-D7 系统

Let us assume that the D3-branes span the directions 0123 and the D7-branes span the directions 01234567. The strings stretched between D3 and D3-branes (also known as the 3-3 strings) give rise to the fields in a four-dimensional gauge theory. The 7-7 strings, on the other hand, give rise to the fields in an eight-dimensional gauge theory that, for the same reason of the fields in the bulk, appear as background fields for the D3-branes.

假设 D3 膜延展开 0123 方向，D7 膜延展开 01234567 方向。拉伸于 D3 膜与 D3 膜之间的弦 (又称 3-3 弦) 产生四维规范理论中的场。另一方面，7-7 弦产生八维规范理论中的场，和 bulk 中的场同理，这些场对 D3 膜而言是背景场。

As we discussed above, the D3-branes by themselves give rise to the 4 d$\mathcal{N} = 4$ SYM. On the other hand, by the precedent argument, we find that the D3-D7- brane system preserves 4 d$\mathcal{N} = 2$ supersymmetry on the D3-worldvolume. We thus expect that the 3-7 strings break $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry. To determine which fields correspond to these strings, we proceed as follows.

如我们上文讨论，单独的 D3 膜会产生 4 d$\mathcal{N} = 4$ 维超对称杨-米尔斯理论。根据前述推导，我们可知 D3-D7 膜系统在 D3 世界体上保留了 4 d$\mathcal{N} = 2$ 超对称。因此我们预期 3-7 弦会将超对称破缺为 $\mathcal{N} = 4$ 到 $\mathcal{N} = 2$ 。我们按下述方式推导哪些场对应这类弦。

We quantize an open string that has a Neumann-Neumann (NN) boundary condition in the directions 0123, a Dirichlet-Neumann (DN) boundary condition in the directions 4567, and a Dirichlet-Dirichlet (DD) boundary condition in the directions 89. The modes of the string are as follows:

我们对开弦进行量子化: 它在 0123 方向满足诺依曼-诺依曼 (NN) 边界条件，在 4567 方向满足狄利克雷-诺依曼 (DN) 边界条件，在 89 方向满足狄利克雷-狄利克雷 (DD) 边界条件。弦的模式如下:

$$X^\mu = x^\mu + p^\mu \tau + \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n}\alpha_n^\mu \left(e^{in(\sigma+\tau)} + e^{in(-\sigma+\tau)}\right)$$

$$\text{DN: } X^\mu = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n}\alpha_n^\mu \left(e^{in(\sigma+\tau)} - e^{in(-\sigma+\tau)}\right)$$

$$\vdots\ X^\mu = v^\mu + (m^\mu - v^\mu)\frac{\sigma}{\pi} + \sum_{n\in\mathbb{Z}-\{0\}} \frac{1}{n}\alpha_n^\mu \left(e^{in(\sigma+\tau)} - e^{in(-\sigma+\tau)}\right)$$

(36)

where $v^\mu$ and $m^v$ are the positions of the D3-brane and D7-brane in the 89 plane, respectively. The fermions have the same modding in the R sector and the opposite one in the NS sector. Let us focus on the NS sector where the space-time bosonic states in the spectrum come from. Recall that each transverse periodic worldsheet boson (respectively, fermion) contributes -1/24 (resp. +1/24) to the zero-point energy, whereas each antiperiodic boson (resp. fermion) contributes +1/48 (resp. -1/48). Suppose that out of the physical eight transverse directions in light-cone quantization, we have $n_{\text{ND}}$ directions. Then, the zero-point energy in the NS sector is

其中 $v^\mu$ 和 $m^v$ 分别是 D3 膜和 D7 膜在 89 平面的位置。费米子在 R 扇区有相同的模化，在 NS 扇区有相反的模化。我们重点关注产生时空玻色态的 NS 扇区。回想一下，世界面上每个横向周期玻色子 (对应费米子) 对零点能贡献-1/24(对应 +1/24)，而每个反周期玻色子 (对应费米子) 贡献 +1/48(对应-1/48)。假设光锥量子化的 8 个物理横向方向中，共有 $n_{\text{ND}}$ 个方向。那么 NS 扇区的零点能为

$$E_0 = (8 - n_{\text{ND}})\left(-\frac{1}{24} - \frac{1}{48}\right) + n_{ND}\left(\frac{1}{48} + \frac{1}{24}\right) = \frac{n_{\text{ND}} - 4}{8}. \tag{37}$$

For $n_{\text{ND}} = 4$ , the zero-point energy $E_0$ in the NS sector is precisely zero. On the contrary to the $3-3$ or $7-7$ strings, whose $E_0 = -1/2$ is cancelled by the operators with weight 1/2 (namely, $\psi^\mu_{-1/2}|0\rangle$ ) carrying the vector indices and giving rise to gauge fields, the $3-7$ strings with $E_0 = 0$ have vertices made from fermionic zero modes. The latter transform in the spinor representations $(\mathbf{2};\mathbf{1}) \oplus (\mathbf{1};\mathbf{2})$ of $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ , which is the symmetries associated with the DN directions 4567. However, only one of the representations, $(\mathbf{2};\mathbf{1})$ or $(\mathbf{1};\mathbf{2})$ , survives the GSO projection. We thus have two states, corresponding to two real scalar fields. Thus, the 3-7 and 7-3 strings give in total four real scalars. These transform in the hypermultiplet representation of the $\mathcal{N} = 2$ supersymmetry in four dimensions.

对于 $n_{\text{ND}} = 4$ ，NS 扇区的零点能 $E_0$ 恰好为零。和 $3-3$ 弦或 $7-7$ 弦不同，这类弦的 $E_0 = -1/2$ 被带权重 1/2 (即 $\psi^\mu_{-1/2}|0\rangle$ ) 的算符抵消，这些算符携带矢量指标并生成规范场，而带 $E_0 = 0$ 的 $3-7$ 弦的顶点由费米零模构成。费米零模属于 $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ 的旋量表示 $(\mathbf{2};\mathbf{1}) \oplus (\mathbf{1};\mathbf{2})$ ，Spin(4) 是和 DN 方向 4567 相关的对称性。但只有一种表示 $(\mathbf{2};\mathbf{1})$ 或 $(\mathbf{1};\mathbf{2})$ 能通过 GSO 投影。因此我们得到两个态，对应两个实标量场。因此 3-7 弦和 7-3 弦总共给出四个实标量。它们属于四维 $\mathcal{N} = 2$ 超对称的超多重态表示。

The fields on the D3-branes can be decomposed into $\mathcal{N} = 2$ multiplets as follows: (1) $A_\mu$ along with the real scalars corresponding to the 89 directions belong to an $\mathcal{N} = 2$ vector multiplet, and (2) the real scalars corresponding to the 4567 directions belong to an $\mathcal{N} = 2$ hypermultiplet. The latter transforms in the adjoint representation of the gauge group.

D3 膜上的场可以分解为 $\mathcal{N}=2$ 多重态，具体如下:(1) $A_\mu$ 与对应 89 方向的实标量共同属于一个 $\mathcal{N}=2$ 矢量多重态，(2) 对应 4567 方向的实标量属于一个 $\mathcal{N}=2$ 超多重态。后者在规范群的伴随表示下变换。

For a system of $N_c$ D3-branes and $N_f$ D7-branes, the worldvolume theory of the D3-branes is therefore the 4 d$\mathcal{N}=2\mathrm{U}(N_c)$ gauge theory with one hypermultiplet in the adjoint representation and $N_f$ hypermultiplets in the fundamental representation.

对于一个包含 $N_c$ 块 D3 膜和 $N_f$ 块 D7 膜的系统，D3 膜的世界体理论因此是 \$ 4 d$\mathcal{N}=2\mathrm{U}(N_c)$ \$ 规范理论，具有一个伴随表示的超多重态和 $N_f$ 个基本表示的超多重态。

Let us make some comments:

我们做一些说明:

1. 4d $\mathcal{N}=2$ gauge theories have a $\mathrm{U}(1)\times\mathrm{SU}(2)R$ -symmetry. The U(1) factor can be realized as a rotational symmetry of the 89 directions, whereas the SU(2) factor can be realized as a subgroup of a rotational symmetry Spin(4) of the 4567 directions.

1. 四维 $\mathcal{N}=2$ 规范理论具有 $\mathrm{U}(1)\times\mathrm{SU}(2)R$ 对称性。U(1) 因子可实现为 89 方向的旋转对称性，而 SU(2) 因子可实现为 4567 方向旋转对称群 Spin(4) 的一个子群。

2. The Coulomb branch of the theory corresponds to the motion of the D3-branes in the 89 directions.

2. 该理论的库仑分支对应 D3 膜在 89 方向上的运动。

3. The 7-7 strings give rise to eight-dimensional gauge fields. From the perspective of the D3-branes, these are non-dynamical background fields. They are realized as the background gauge fields associated with the flavor symmetry $\mathrm{U}\left(N_f\right)$ . Indeed, the fields corresponding to the 3-7 strings carry two indices associated with the Chan-Paton charges, $\phi_{iA}$ where $i=1,\ldots,N_c$ is the index associated with the D3-branes and corresponds to the fundamental representation of the $\mathrm{U}(N_c)$ gauge group, and $A=1,\ldots,N_f$ is the index associated with the D7- branes and corresponds to the flavor symmetry.

3. 7-7 弦产生八维规范场。从 D3 膜的角度来看，这些都是非动力学背景场，它们实现为与味对称性 $\mathrm{U}\left(N_f\right)$ 关联的背景规范场。的确，对应 3-7 弦的场带有两个与陈-帕顿荷关联的指标，$\phi_{iA}$ 其中 $i=1,\ldots,N_c$ 是与 D3 膜关联的指标，对应 $\mathrm{U}(N_c)$ 规范群的基本表示，$A=1,\ldots,N_f$ 是与 D7 膜关联的指标，对应味对称性。

4. As the bulk fields, the VEVs of the scalar fields on the D7-branes become parameters for the D3-branes. In this case, the fields corresponding to the 89 directions transverse to the D7-branes become the mass parameters of the hypermultiplet $\phi_{iA}$ . In fact, the mass of the 3-7 strings is proportional to the relative distance in the coordinates DD between D3 and D7-branes:

4. 作为体场，D7 膜上标量场的真空期望值成为 D3 膜的参数。在这种情况下，垂直于 D7 膜的 89 方向对应的场成为超多重态 $\phi_{iA}$ 的质量参数。事实上，3-7 弦的质量正比于 D3 膜与 D7 膜在坐标 DD 中的相对距离:

$$x_{\mathrm{D7}}^{89} - x_{\mathrm{D3}}^{89} = m - v \tag{38}$$

where $v$ is the position of the stack of D3-branes and corresponds to the VEV of the complex scalar of the $\mathrm{U}(1)$ part of the $\mathrm{U}(N_c)$ gauge group. This is a standard formula for $\mathcal{N} = 2$ supersymmetry, as also explained below.

其中 $v$ 是 D3 膜叠的位置，对应 $\mathrm{U}(N_c)$ 规范群 $\mathrm{U}(1)$ 部分复标量的真空期望值。这是 $\mathcal{N} = 2$ 超对称的标准公式，下文也会说明。

5. The brane configuration provides an explicit realization of a useful trick in obtaining the mass of the $\mathcal{N} = 2$ hypermultiplets as VEVs of the background vector supermultiplet associated with the flavor symmetry. In $\mathcal{N} = 1$ superfield notation, the $\mathcal{N} = 2$ vector multiplet decomposes into a gaugino superfields

5. 该膜构型给出了一个实用技巧的具体实现: 将 $\mathcal{N} = 2$ 超多重态的质量取为与味对称性关联的背景矢量超多重态的真空期望值。在 $\mathcal{N} = 1$ 超场记号中，$\mathcal{N} = 2$ 矢量多重态可分解为戈金诺超场

$W_\alpha$ and a chiral superfield $\varphi$ , while the hypermultiplets decompose into pairs of chiral superfields $(Q_A, \widetilde{Q}_A)$ . The D3-brane action reads

$W_\alpha$ 和一个手征超场 $\varphi$ ，而超多重态分解为成对的手征超场 $(Q_A, \widetilde{Q}_A)$ 。D3 膜的作用量为

$$\sum_A \int d\theta^2 \widetilde{Q_A} \varphi Q_A + \frac{1}{g^2} \int d\theta^2 W_\alpha^2 + \sum_A \int d\theta^2 d\overline{\theta}^2 \left( Q_A^\dagger e^V Q_A + \widetilde{Q}_A^\dagger e^{-V} \widetilde{Q}_A \right).$$

(39)

Adding the contribution of the vector multiplets of the D7-branes containing gaugino superfields $W_\alpha^F$ and the scalar partner $\varphi^F$ , we obtain

加入包含戈金诺超场 $W_\alpha^F$ 和标量伙伴 $\varphi^F$ 的 D7 膜矢量多重态的贡献，我们得到

$$\int d\theta^2 \widetilde{Q} \varphi Q + \frac{1}{g^2} \int d\theta^2 W_\alpha^2 + \int d\theta^2 d\overline{\theta}^2 \left( Q^\dagger e^{V+V_F} Q + \widetilde{Q}^\dagger e^{-V-V_F} \widetilde{Q} \right)$$
$$+ \int d\theta^2 \widetilde{Q} \varphi^F Q + \frac{1}{g_F^2} \int d\theta^2 (W_\alpha^F)^2,$$

(40)

where now we consider $Q$ as a matrix acted upon by the gauge and flavor group. As seen by the D3-branes, the gauge coupling of the D7-branes is effectively zero, and the D7 scalar fields are frozen to their VEV. Sending $g_F \to 0$ with $\langle \varphi^F \rangle \neq 0$ , we obtain the mass terms:

此处我们将 $Q$ 视为规范群与味群作用的矩阵。对 D3 膜而言，D7 膜的规范耦合实际上为零，且 D7 的标量场被冻结在其真空期望值处。令 $g_F \to 0$ 结合 $\langle \varphi^F \rangle \neq 0$ ，我们得到质量项:

$$\int \widetilde{Q} \varphi^F Q \to \int m_A \widetilde{Q}^A Q^A \tag{41}$$

We see that the mass parameters, which are complex, can be viewed as background fields residing in the flavor vector multiplet. There is an extra contribution to the mass from the complex scalar $\varphi$ in the abelian part of the D3-brane gauge group. In fact, let us consider the terms $\widetilde{Q}_A \varphi Q_A + m_A \widetilde{Q}_A Q_A$ . If the VEV of $\varphi$ takes the diagonal form, say $\langle \phi_{ij} \rangle = -v\delta_{ij}$ , we obtain the mass $m_A - v$ for the $A$ -th hypermultiplet, which corresponds to the relative distance between D3- and D7-branes.

我们可以看到，这些复质量参数可以被视作位于味道矢量多重态中的背景场。D3 膜规范群的阿贝尔部分中的复标量 $\varphi$ 会对质量产生一项额外贡献。实际上，我们来考虑项 $\widetilde{Q}_A \varphi Q_A + m_A \widetilde{Q}_A Q_A$ 。如果 $\varphi$ 的真空期望值取对角形式，例如 $\langle \phi_{ij} \rangle = -v\delta_{ij}$ ，我们就会得到第 $A$ 个超多重子的质量 $m_A - v$ ，该质量对应 D3 膜与 D7 膜之间的相对距离。

## Other Gauge Groups

## 其他规范群

Let us turn to the gauge groups that are different from $\mathrm{U}(N)$ . In type I string theory, we know how to construct the gauge group of the orthogonal-type. In general, in type I′ string theory, every Dp-brane is always accompanied by an orientifold O$p$ - plane. We recall that the O$p$ -plane corresponds to the combination of the worldsheet parity Ω and the $\mathbb{Z}_2$ reflection in the directions transverse to the plane. In general, Ω can give either orthogonal- or symplectic-type gauge groups, depending on the matrices used in the projection, and the corresponding orientifold planes are called $\mathrm{Op}^{\mp}$ , respectively. In type I string theory, cancellation of anomalies, which states that the sum of the charges of the branes and orientifold planes has to be zero in a compact space, imposes that the gauge group has to be of the orthogonal-type. However, in our case, we are dealing with a non-compact space, and so we can have both orthogonal- and symplectic-type gauge groups. An important rule is as follows: If we choose the group on the brane parallel to the orientifold plane to be of one of these types, then the gauge group on brane with the four DN coordinates is projected to the other type [18]. Let us give some examples of the system of $N_c$ physical D3-branes and $N_f$ physical D7-branes on top of the $\mathrm{O7}^{\pm}$ -plane. The global and gauge symmetry algebras and the matter fields are as follows:

下面我们讨论不同于 $\mathrm{U}(N)$ 的规范群。在 I 型弦论中，我们已知如何构造正交型规范群。一般而言，在 I′ 型弦论中，每个 Dp 膜总是伴随一个 O O$p$ 平面。我们回顾: O$p$ 平面对应世界面 parity Ω 与垂直于该平面方向上的 $\mathbb{Z}_2$ 反射的组合。一般来说，根据投影所用矩阵的不同，Ω 可以给出正交型或辛型规范群，对应的定向模平面分别称为 $\mathrm{Op}^{\mp}$ 。在 I 型弦论中，反常抵消要求紧致空间中膜与定向模平面的总电荷为零，这要求规范群必须是正交型的。但本文我们讨论的是非紧致空间，因此正交型和辛型规范群都可以存在。一条重要规则如下: 如果我们取平行于定向模平面的膜上的规范群为其中一种类型，那么带有四个 DN 坐标的膜上的规范群会被投影为另一种类型 [18]。下面我们给出 $N_c$ 物理 D3 膜与 $N_f$ 物理 D7 膜位于 $\mathrm{O7}^{\pm}$ 平面上方的系统实例。整体对称代数、规范对称代数与物质场如下:

(42)

| Type of O7-plane | Global | Gauge | Matter fields |
|---|---|---|---|
| O7- | $\mathfrak{so}\left(2N_f\right)$ | usp(2 $N_f$ ) | 1A and $N_f F$ |
| O7+ | $\mathfrak{usp}\left(2N_f\right)$ | $\mathfrak{so}\left(2N_f\right)$ | 1S and $N_f F$ |

where $A, S$ , and $F$ denote hypermultiplets in the second-rank antisymmetric, second-rank symmetric, and fundamental representations, respectively. Observe that the numbers $2N_c$ and $2N_f$ are always even, since the branes are accompanied by their images. The fact that the symplectic-type gauge group always comes with the orthogonal-type flavor symmetry algebra (and vice-versa) is a known fact in $\mathcal{N} = 2$ supersymmetry.

其中 $A, S$ 和 $F$ 分别表示二阶反对称表示、二阶对称表示和基础表示中的超多重子。注意到 $2N_c$ 和 $2N_f$ 始终为偶数，这是因为膜总是伴随其镜像膜存在。辛型规范群始终伴随正交型味对称代数 (反之亦然) 这一结论是 $\mathcal{N} = 2$ 超对称中的已知结论。

## Orbifolds

## 轨形

Another way of realizing $\mathcal{N} = 2$ gauge theory is to consider $N$ D3-branes sitting at the singularity of the orbifold $\mathbb{R}^4/\Gamma \times \mathbb{R}^2$ , where $\Gamma$ is a discrete group of an SU (2) factor of the isometry group Spin (4) $\cong$ SU (2) $\times$ SU (2) of $\mathbb{R}^4$ . The projection breaks the isometry of the transverse space (and the R-symmetry of the corresponding gauge theory) as Spin (6) $\to$ SU (2) $\times$ U (1) and the supersymmetry as $\mathcal{N} = 4 \to \mathcal{N} = 2$ .

实现 $\mathcal{N} = 2$ 规范理论的另一种方法，是考虑放置在轨形 $\mathbb{R}^4/\Gamma \times \mathbb{R}^2$ 奇点处的 $N$ 张 D3 膜，其中 $\Gamma$ 是 $\mathbb{R}^4$ 等距群 Spin (4) $\cong$ SU (2) $\times$ SU (2) 的 SU (2) 因子的离散群。该投影破坏了横空间的等距性 (以及对应规范理论的 R 对称性) 为 Spin (6) $\to$ SU (2) $\times$ U (1) ，并将超对称性破坏为 $\mathcal{N} = 4 \to \mathcal{N} = 2$ 。

To determine the gauge theory living on branes on $\mathbb{R}^6/\Gamma$ [19], we have to study the action of the orbifold projection on the worldvolume fields. For simplicity, we consider only abelian groups $\mathbb{Z}_k$ . In the covering space $\mathbb{R}^6$ , a D3-brane has $k-1$ images under $\mathbb{Z}_k$ . We can think of a collection of $k$ D3-branes as making a physical D3-brane. $\mathbb{Z}_k$ acts on the set of $k$ -branes by a cyclic permutation: this is called the regular representation of $\Gamma$ . Before the projection, a set of $kN$ -branes realizes a $U(kN)$ gauge theory. Let each element $\alpha \in \Gamma$ act on the Chan-Paton factors with a matrix $\gamma_\alpha$ in the regular representation of $\Gamma$ . The bosonic fields are projected by

为了确定 $\mathbb{R}^6/\Gamma$ 上膜世界 volume 上的规范理论 [19]，我们必须研究轨形投影在世界 volume 场上的作用。为简化起见，我们仅考虑阿贝尔群 $\mathbb{Z}_k$ 。在覆盖空间 $\mathbb{R}^6$ 中，一张 D3 膜在 $\mathbb{Z}_k$ 作用下有 $k-1$ 个像。我们可以将一组 $k$ 张 D3 膜看作一张物理 D3 膜。$\mathbb{Z}_k$ 通过循环置换作用在这组 $k$ 膜上: 这被称为 $\Gamma$ 的正则表示。投影前，一组 $kN$ 膜实现了 $U(kN)$ 规范理论。令每个元素 $\alpha \in \Gamma$ 通过矩阵 $\gamma_\alpha$ (取 $\Gamma$ 正则表示下) 作用在陈-帕顿因子上。玻色场被投影为

$$A_\mu = \gamma_\alpha A_\mu \gamma_\alpha^{-1}$$

$$\phi^a = R(\alpha)^{ab} \gamma_\alpha \phi^b \gamma_\alpha^{-1}, \tag{43}$$

where $a, b = 1, \ldots, 4$ . The matrices $R(\alpha)$ take into account that the original $\mathcal{N} = 4$ scalars transform under Spin(6) $\cong$ SU(4) (in the vector representation) and therefore under its subgroup $\Gamma$ .

其中 $a, b = 1, \ldots, 4$ 。矩阵 $R(\alpha)$ 说明了原本的 $\mathcal{N} = 4$ 标量在 Spin(6) $\cong$ SU(4) 的 (向量表示下) 变换，因此也在其子群 $\Gamma$ 下变换。

As an example, consider the orbifold $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^2$ . The action of $\mathbb{Z}_2$ is given by

举一个例子，考虑轨形 $\mathbb{R}^4/\mathbb{Z}_2 \times \mathbb{R}^2$ 。$\mathbb{Z}_2$ 的作用由下式给出

$$\phi^a \to -\phi^a,\ a = 1, 2, 3, 4,\ \phi^a \to \phi^a,\ a = 5, 6. \tag{44}$$

There is only one nontrivial matrix $\gamma_\alpha$ corresponding to the generator of $\mathbb{Z}_2$ , and it can be chosen as $\gamma_\alpha = \operatorname{diag}\{I_N, -I_N\}$ . A simple application of the previous rules shows that the gauge group is $\mathrm{U}(N) \times \mathrm{U}(N)$ , with adjoint $\mathcal{N} = 2$ vector multiplets and two bifundamental hypermultiplets.

对应 $\mathbb{Z}_2$ 生成元只有一个非平凡矩阵 $\gamma_\alpha$ ，它可以选为 $\gamma_\alpha = \operatorname{diag}\{I_N, -I_N\}$ 。对上述规则的简单应用可知，规范群为 $\mathrm{U}(N) \times \mathrm{U}(N)$ ，伴随表示下有 $\mathcal{N} = 2$ 个向量多重态和两个双基本超多重态。

The gauge theories obtained as projections have a characteristic quiver (or moose) form. In the $\mathcal{N} = 2$ case, a complete classification exists [19] based on the McKay correspondence [20]. The discrete subgroups of $SU(2)$ are in one-toone correspondence with the simply laced Lie algebras $A_k, D_k$ and $E_6, E_7, E_8$ . The gauge theory on $N$ physical branes at a singularity $\mathbb{R}^4/\Gamma \times \mathbb{R}^2$ is associated with the affine Dynkin diagram of the Lie algebra corresponding to $\Gamma$ . A $U(n_i N)$ vector multiplet is associated with each node with Dynkin label $n_i$ , and a bifundamental hypermultiplet is associated with each link connecting two different nodes. For $N = 1$ , the Higgs branch of the theory coincides with the hyperKähler manifold $\mathbb{R}^4/\Gamma$ , corresponding to the geometry probed by the D3-brane. And indeed, the very same quiver was used by Kronheimer [21] to provide an hyperKähler quotient construction of the singularity $\mathbb{R}^4/\Gamma$ and its resolution, the so-called ALE (asymptotically locally Euclidean) spaces [22-25]. The singularity is obtained as the space of solutions of (34), and the ALE spaces are obtained using FI parameters $\zeta \neq 0$ as in (35).

通过投影得到的规范理论具有特征性的箭图 (或称莫尔斯) 形式。在 $\mathcal{N} = 2$ 情形中，基于麦凯对应 [19] 已经存在完整分类 [20]。$SU(2)$ 的离散子群与单李代数 $A_k, D_k$ 和 $E_6, E_7, E_8$ 一一对应。位于奇点 $\mathbb{R}^4/\Gamma \times \mathbb{R}^2$ 处的 $N$ 物理膜上的规范理论对应于对应 $\Gamma$ 的李代数的仿射 Dynkin 图。每个带有 Dynkin 标记 $n_i$ 的节点对应一个 $U(n_i N)$ 向量多重态，连接两个不同节点的每条边对应一个双基本超多重态。对 $N = 1$ 而言，该理论的希格斯分支与超凯勒流形 $\mathbb{R}^4/\Gamma$ 重合，对应 D3 膜探测的几何。事实上，Kronheimer[21] 早已使用完全相同的箭图给出了奇点 $\mathbb{R}^4/\Gamma$ 及其解消的超凯勒商构造，即所谓的 ALE(渐近局部欧几里得) 空间 [22-25]。奇点由 (34) 的解空间得到，而 ALE 空间则如 (35) 所示利用 FI 参数 $\zeta \neq 0$ 构造得到。

A $d+1$ -dimensional theory with eight supercharges with the same gauge and matter content is realized on the worldvolume of Dp-branes at a $\mathbb{R}^4/\Gamma \times \mathbb{R}^{5-p}$ singularity.

具有 8 个超荷、规范与物质内容相同的 $d+1$ 维理论，可以实现为 $\mathbb{R}^4/\Gamma \times \mathbb{R}^{5-p}$ 奇点处 Dp 膜世界 volume 上的理论。

## Other Realizations

## 其他实现方法

Another method to obtain $\mathcal{N} = 2$ gauge theories is to consider, for example, the D4- branes stretched between two parallel NS5-branes. We will discuss this technique later in section "Hanany-Witten Brane Configurations in Four Dimensions."

获得 $\mathcal{N} = 2$ 规范理论的另一种方法，例如可以是考虑伸展在两个平行 NS5-膜之间的 D4-膜。我们会在后续“四维中的 Hanany-Witten 膜构造”一节讨论这个方法。

## $\mathcal{N} = 4$ Gauge Theories in Three Dimensions

## $\mathcal{N} = 4$ 维规范理论在三维中的情形

These theories have eight real supercharges, which is the same amount of supersym-metry as for $\mathcal{N} = 2$ gauge theories in four dimensions. They can be realized in a simple way, for example, on the worldvolume of D2-branes in the system consisting of D2- and D6-branes in type IIA string theory. We may obtain this configuration by T-dualizing the aforementioned D3-D7 system, for example, along the direction (3).

这些理论拥有 8 个实超荷，超对称数量和四维中的 $\mathcal{N} = 2$ 规范理论相同。它们可以通过简单的方式实现，例如 IIA 型弦论中由 D2 膜和 D6 膜构成的系统里，D2 膜的世界体积上就可以实现这类理论。举例来说，我们可以将前文提到的 D3-D7 系统沿方向 (3) 做 T 对偶得到该构型。

A 3 d$\mathcal{N} = 4$ gauge theory can be realized by dimensionally reducing a 6 d $\mathcal{N} = (1, 0)$ to three dimensions. The 3 d theory has $R$ -symmetry $\mathrm{SU}(2)_H \times \mathrm{SU}(2)_V$ , where $\mathrm{SU}(2)_H$ can be identified as the SU (2) $R$ -symmetry of the 6 d theory and $\mathrm{SU}(2)_V$ corresponds to the rotational symmetry in the three directions used in going to three dimensions.

3 d$\mathcal{N} = 4$ 规范理论可以通过将 6 d $\mathcal{N} = (1, 0)$ 维约化到三维得到。3 d 理论具有 $R$ 对称性 $\mathrm{SU}(2)_H \times \mathrm{SU}(2)_V$，其中 $\mathrm{SU}(2)_H$ 可以等同于 6 d 理论的 SU (2) $R$ 对称性，而 $\mathrm{SU}(2)_V$ 对应降维到三维过程中所用的三个方向的旋转对称性。

There are two types of multiplets for the matter fields. The first one is the hypermultiplet. As in 4d, it contains four scalars that transform only under the $\mathrm{SU}(2)_H$ symmetry. The second one is the vector multiplet. This consists of a gauge field $A_\mu$ and three real scalars. The scalars transform as a triplet under $\mathrm{SU}(2)_V$ . Moreover, in three dimensions, the gauge field $A_\mu$ is dual to a scalar $a$ via $F^{\mu\upsilon} = \varepsilon^{\mu\upsilon\tau}\partial_\tau a$ . For an abelian group, such a transformation is perfectly legitimate, and the multiplet can be represented by four scalars. The moduli space consists of a Coulomb branch, where the scalars in the vector multiplet and the dual photon $a$ have a VEV, and of an Higgs branch, where the hypermultiplet scalars have a VEV. In addition, there could exist mixed branches. $\mathcal{N} = 4$ supersymmetry requires both Coulomb and Higgs branches to be hyperKähler manifolds.

物质场存在两类多重态。第一类是超多重态。和四维情形一样，它包含四个仅在 $\mathrm{SU}(2)_H$ 对称性下变换的标量。第二类是矢量多重态，由一个规范场 $A_\mu$ 和三个实标量构成。这些标量在 $\mathrm{SU}(2)_V$ 下按三重态变换。此外，在三维中，规范场 $A_\mu$ 可以通过 $F^{\mu\upsilon} = \varepsilon^{\mu\upsilon\tau}\partial_\tau a$ 对偶为一个标量 $a$。对于阿贝尔群，这类变换是完全合理的，因此该多重态可以用四个标量表示。该理论的模空间由库仑分支和希格斯分支构成，库仑分支上矢量多重态的标量和对偶光子 $a$ 获得真空期望值，希格斯分支上超多重态的标量获得真空期望值，此外还可能存在混合分支。$\mathcal{N} = 4$ 超对称要求库仑分支和希格斯分支都是超凯勒流形。

There are some important theorems:

这里有若干重要定理:

- The Higgs branch is not renormalized by quantum effects. The structure of the Higgs branch is exact at the classical level.

- 希格斯分支不会被量子效应重整化。希格斯分支的结构在经典水平就是精确的。

- The Higgs branch does not depend on the masses of the hypermultiplets, but the Coulomb branch does.

- 希格斯分支不依赖于超多重态的质量，但库仑分支依赖。

The first statement follows from the fact that the gauge coupling can be promoted to a vector multiplet background superfield and scalars in different multiplets cannot mix [12,13]. As a result, the Higgs branch does not get renormalized. Similarly, the second statement follows from the fact that mass terms can be regarded as VEVS of scalars in a background flavor symmetry vector multiplet, and, as such, they transform as $(\mathbf{3}, \mathbf{1})$ of $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_H$ . We see that the mass of a hypermultiplet consists of three real numbers.

第一个论断源于如下事实: 规范耦合可以提升为矢量多重态的背景超场，且不同多重态中的标量无法混合 [12,13]，因此希格斯分支不会得到重整化。类似地，第二个论断源于: 质量项可以看作背景味对称性矢量多重态中标量的真空期望值，因此它们按照 $\mathrm{SU}(2)_V \times \mathrm{SU}(2)_H$ 的 $(\mathbf{3}, \mathbf{1})$ 变换，我们可以看到一个超多重态的质量由三个实数构成。

## Realization on the Branes

## 膜上实现

If we take a system consisting of $N_c$ D2-branes spanning the directions (012) and $N_f$ D6-branes spanning (0123456), the theory on the D2-branes is the 3 d$\mathcal{N} = 4$ U $(N_c)$ gauge theory with one hypermultiplet in the adjoint representation and $N_f$ hypermultiplets in the fundamental representations.

如果我们取一个由覆盖方向 (012) 的 $N_c$ 张 D2 膜和覆盖方向 (0123456) 的 $N_f$ 张 D6 膜组成的系统，D2 膜上的理论就是 3 d$\mathcal{N} = 4$ U $(N_c)$ 规范理论，它包含 1 个伴随表示的超多重子和 $N_f$ 个基本表示的超多重子。

One can generalize this by taking an orientifold $O^{6-}$ -plane parallel to the D6- branes. When the latter is on top of the former, this gives rise to the gauge algebra $\mathfrak{so}\left(2N_f\right)$ , where $2N_f$ are the number of D6-branes plus their images. The gauge algebra realized on the D2-brane is $\mathfrak{usp}\,(2N_c)$ . As before, the 2-6 strings can be regarded as the hypermultiplets in the fundamental representation of $\mathfrak{usp}\,(2N_c)$ . This fixes the form of the gauge group to be $\mathrm{USp}\,(2N_c)$ , not $\mathrm{USp}\,(2N_c)\,/\mathbb{Z}_2$ , since they transform nontrivially under the $\mathbb{Z}_2$ center of $\mathrm{USp}\,(2N_c)$ . This system of branes thus realizes the 3 d$\mathcal{N}=4\,\mathrm{USp}\,(2N_c)$ gauge group, one hypermultiplet in the adjoint representation, and $N_f$ hypermultiplets in the fundamental representation. In the special case of $N_c=2$ , we have $\mathrm{USp}\,(2)\cong\mathrm{SU}\,(2)$ , with an obvious $\mathfrak{so}\left(2N_f\right)$ flavor symmetry algebra.

我们可以通过引入一个平行于 D6 膜的 O $O^{6-}$ 定向亏面对这个系统进行推广。当 D6 膜与 O6 平面重合时，就会得到规范代数 $\mathfrak{so}\left(2N_f\right)$ ，其中 $2N_f$ 是 D6 膜的数量加上它们的镜像的数量。D2 膜上实现的规范代数为 $\mathfrak{usp}\,(2N_c)$ 。和之前一样，2-6 弦可以看作 $\mathfrak{usp}\,(2N_c)$ 基本表示下的超多重子。这确定了规范群的形式为 $\mathrm{USp}\,(2N_c)$ 而非 $\mathrm{USp}\,(2N_c)\,/\mathbb{Z}_2$ ，因为它们在 $\mathrm{USp}\,(2N_c)$ 的 $\mathbb{Z}_2$ 中心下具有非平凡变换。因此这个膜系统实现了 3 d$\mathcal{N}=4\,\mathrm{USp}\,(2N_c)$ 规范群，包含 1 个伴随表示的超多重子和 $N_f$ 个基本表示的超多重子。在 $N_c=2$ 的特殊情况下，我们得到 $\mathrm{USp}\,(2)\cong\mathrm{SU}\,(2)$ ，它具有明显的 $\mathfrak{so}\left(2N_f\right)$ 味对称代数。

## The Higgs Branch

## 希格斯分支

The Higgs branch exists only when the hypermultiplets are massless, in which case the D2-branes must be in the same point in the (789) directions as the D6- branes. The position of the D6-branes in such directions is the mass $\mathbf{m}$ of the hypermultiplets. As we emphasized above, $\mathbf{m}$ consists of three real numbers. On the other hand, the position of the D2-branes in such directions is the value $x$ of the three real scalars that are partners of the gauge fields. In order to go along the Higgs branch, every mass $\mathbf{m}$ has to be equal, and the mass of the fields, $\mathbf{m}-\mathbf{x}$ , has to be zero; in other words, the D2-branes and D6-branes must be on top of each other.

希格斯分支仅在超多重态为无质量时存在，此时 D2 膜必须与 D6 膜处于 (789) 方向的同一点。D6 膜在这些方向上的位置就是超多重态的质量 $\mathbf{m}$ 。正如我们上文强调的，$\mathbf{m}$ 由三个实数构成。另一方面，D2 膜在这些方向上的位置就是规范场伴生的三个实标量的取值 $x$ 。要沿希格斯分支移动，所有质量 $\mathbf{m}$ 必须相等，场的质量 $\mathbf{m}-\mathbf{x}$ 必须为零；换句话说，D2 膜与 D6 膜必须互相重叠。

When this is the case, the D2-brane becomes a point inside the D6-brane, and it appears as an instanton for the gauge field on the D6-brane [8]. In fact, the coupling $C_3\wedge F\wedge F$ on the D6-branes (see (13)) contributes to the factor $F\wedge F$ upon the equation of motion for $C_3$ :

满足该条件时，D2 膜成为 D6 膜内部的一个点，它对应 D6 膜上规范场的一个瞬子 [8]。实际上，D6 膜上的耦合 $C_3\wedge F\wedge F$ (见 (13)) 对 $C_3$ 的运动方程贡献出因子 $F\wedge F$ :

$$d*F_4=(F\wedge F)\,\delta^{(789)}\,(\mathbf{x})\,. \tag{45}$$

This means that an instanton with topological number $k=\int F\wedge F$ , where the fields take values in the directions (4567), is the source of $k$ units of charge for $C_3$ . Also, the D2-brane is a source for $C_3$ , and so $k$ D2-branes inside the D6-branes can be exchanged for the instanton with topological number $k$ . A D2-brane touching the D6-brane in a point, with zero VEVs for the hypermultiplets, is a singular instanton with zero size. When the VEVs for the hypermultiplets are turned on, the Higgs branch becomes identical to the moduli space of $N_c$ instantons of the gauge group $\mathrm{U}\left(N_f\right)$ , where $N_c$ and $N_f$ are the number of D2- and D6-branes, respectively.

这意味着，拓扑数为 $k=\int F \wedge F$ 、场取值在 (4567) 方向的瞬子，是 $C_3$ 携带 $k$ 单位电荷的源。此外，D2 膜本身就是 $C_3$ 的源，因此 D6 膜内部的 $k$ 个 D2 膜可以等价为拓扑数 $k$ 的瞬子。D2 膜在一点接触 D6 膜、超多重态的真空期望值为零的情况，对应零尺寸的奇异瞬子。当开启超多重态的真空期望值后，希格斯分支就等同于规范群 $\mathrm{U}\left(N_f\right)$ 下 $N_c$ 瞬子的模空间，其中 $N_c$ 和 $N_f$ 分别是 D2 膜和 D6 膜的数量。

In field theory, the Higgs branch is obtained by setting to zero the triplets of D-terms of the gauge theory and modding by the gauge group action, without including quantum corrections. Remarkably, this is precisely the ADHM construction [26] of the instanton moduli space which has been known for 20 years before the discovery of D-branes. Therefore, the Higgs branch of the $\mathrm{U}(N_c)$ gauge theory with one adjoint hypermultiplet and $N_f$ fundamental hypermultiplets provides the realization of the moduli space of $\mathrm{U}\left(N_f\right)$ instantons. Similarly, the Higgs branch of the $\mathrm{USp}(N_c)$ gauge theory with one adjoint hypermultiplet and $N_f$ fundamental hypermultiplets is identical to the moduli space of $\mathrm{SO}\left(2N_f\right)$ instantons. This realization also provides the ADHM construction of the latter.

在场论中，希格斯分支是通过将规范理论的 D 项三元组置零，再商去规范群作用得到的，不包含量子修正。值得注意的是，这正是 D 膜发现前 20 年就已提出的 ADHM 瞬子模空间构造 [26]。因此，带有一个伴随超多重态和 $N_f$ 个基本超多重态的 $\mathrm{U}(N_c)$ 规范理论，其希格斯分支实现了 $\mathrm{U}\left(N_f\right)$ 瞬子的模空间。同理，带有一个伴随超多重态和 $N_f$ 个基本超多重态的 $\mathrm{USp}(N_c)$ 规范理论，其希格斯分支等同于 $\mathrm{SO}\left(2N_f\right)$ 瞬子的模空间。这一实现也给出了后者的 ADHM 构造。

To understand what happens at the singular point of this moduli space, i.e., when the size of the instanton becomes zero (also known as the small instanton), was an open problem. From the point of view of the D6-brane, the instanton, which corresponds to the D2-brane, becomes a point, and it can now leave the D6-brane and move in the (789) directions [8,27].

理解该模空间奇异点 (即瞬子尺寸变为零，也称为小瞬子) 处的物理曾是一个开放问题。从 D6 膜的视角来看，对应 D2 膜的瞬子成为一个点，它可以离开 D6 膜，在 (789) 方向运动 [8,27]。

## The Coulomb Branch

## 库仑分支

This branch of the moduli space is much more complicated than the Higgs branch, since it receives quantum corrections both perturbatively and non-perturbatively. We consider, for simplicity, two cases that have a simple D2-D6-brane interpretation: the 3 d$\mathcal{N}=4\mathrm{U}(1)$ gauge theory with $N_f$ hypermultiplets of charge 1 and the SU(2) gauge theory with $N_f$ hypermultiplets in the fundamental representation. Note that in each case the adjoint and antisymmetric hypermultiplets of the D2-D6 system decouple.

模空间的这个分支比希格斯分支复杂得多，因为它同时受到微扰和非微扰量子修正。为简化讨论，我们考虑两种可由 D2-D6 膜给出简单解释的情形: 带 $N_f$ 个电荷 1 超多重态的 3 d$\mathcal{N}=4$U(1) 规范理论，以及基本表示中含 $N_f$ 个超多重态的 SU(2) 规范理论。注意两种情形下 D2-D6 系统的伴随和反对称超多重态都会退耦。

Let us consider the case where the VEVs of the triplet $\varphi$ of real scalar fields in the vector multiplet are nonzero. For SU(2), using a gauge transformation, we can bring them to the diagonal form $\varphi=\operatorname{diag}(\varphi,-\varphi)$ belonging to the Cartan subalgebra. There remains a residual discrete symmetry corresponding to the $\mathbb{Z}_2$ Weyl transformation, where $\varphi \rightarrow -\varphi$ . For $\varphi \neq 0$ , the unbroken gauge group is abelian, and the photon can be dualized to a real scalar field $a$ , as discussed above. The $\mathbb{Z}_2$ Weyl group also acts on $a$ as $a \rightarrow -a$ .

我们考虑矢量多重态中三重实标量场 $\varphi$ 获得非零真空期望值的情况。对 SU(2)，通过规范变换我们可以将其对角化为属于嘉当子代数的 $\varphi=\operatorname{diag}(\varphi,-\varphi)$ ，此时还剩下对应 $\mathbb{Z}_2$ 外尔变换的剩余离散对称性，其中 $\varphi \rightarrow -\varphi$。对 $\varphi \neq 0$ ，未破缺的规范群是阿贝尔群，光子可以对偶化为一个实标量场 $a$ ，正如之前讨论的那样。$\mathbb{Z}_2$ 外尔群也按照 $a \rightarrow -a$ 作用在 $a$ 上。

The classical metric is

经典度规为

$$
d s^2=g^{-2} d \varphi^2+g^2 d a^2, \tag{46}
$$

where the second term can be realized as the dualization of $g^{-2} F_{\mu v} F^{\mu v}$ as follows. Consider the terms $g^{-2} F_{\mu v} F^{\mu v} / 2+a \varepsilon^{\mu v \tau} \partial_\mu F_{v \tau}$ , where $a$ can be regarded as a Lagrange multiplier for the condition $\partial_{\{\mu} F_{v \tau\}}=0$ . The equations of motion of $F_{\mu v}$ are $F_{\mu v}=g^2 \varepsilon_{\mu v \tau} \partial^\tau a$ . Hence, the terms $g^{-2} F_{\mu v} F^{\mu v} / 2$ become $g^2 d a^2$ as required. It should be emphasized that $a$ is a compact real scalar with radius $g$ . The VEVs of the triplet $\boldsymbol{\varphi}$ of real scalar fields together with that of the compact scalar $a$ parametrize the moduli space $\mathbb{R}^3 \times S^1$ in case of the U(1) gauge theory and $\left(\mathbb{R}^3 \times S^1\right) / \mathbb{Z}_2$ in case of the SU (2) gauge theory.

其中第二项可通过 $g^{-2} F_{\mu v} F^{\mu v}$ 的对偶化按如下方式实现: 考虑项 $g^{-2} F_{\mu v} F^{\mu v} / 2+a \varepsilon^{\mu v \tau} \partial_\mu F_{v \tau}$ ，此处 $a$ 可被视为约束条件 $\partial_{\{\mu} F_{v \tau\}}=0$ 的拉格朗日乘子。$F_{\mu v}$ 的运动方程为 $F_{\mu v}=g^2 \varepsilon_{\mu v \tau} \partial^\tau a$ 。因此项 $g^{-2} F_{\mu v} F^{\mu v} / 2$ 变为我们需要的 $g^2 d a^2$ 。需要强调的是，$a$ 是半径为 $g$ 的紧致实标量。三重实标量 $\boldsymbol{\varphi}$ 的真空期望值加上紧致标量 $a$ 的真空期望值，在 U(1) 规范理论下参数化模空间 $\mathbb{R}^3 \times S^1$ ，在 SU (2) 规范理论下参数化模空间 $\left(\mathbb{R}^3 \times S^1\right) / \mathbb{Z}_2$ 。

The quantum corrections can change the topology of this space and produce terms of the form

量子修正可以改变这个空间的拓扑，并产生如下形式的项

$$
d s^2=g^{-2}(\varphi)(d \varphi)^2+g^2(\varphi)\left(d a+w_i(\varphi) d \varphi^i\right)^2 \tag{47}
$$

For $\mathcal{N}=4$ supersymmetric theories, the metric has to be hyperKähler, and this imposes relations among the functions $g$ and $w_i$ :

对于 $\mathcal{N}=4$ 超对称理论，度规必须是超凯勒的，这给函数 $g$ 和 $w_i$ 加上了约束:

- Perturbatively, as in four dimensions, the quantum corrections occur only at 1- loop. An explicit calculation or geometric arguments [28,29] show that the field $w$ in (47) is essentially the Dirac monopole. If $\varphi = (\phi, \theta, \varphi)$ , then $w_i d\varphi^i = \; s\cos\theta d\phi/2$ , where the integer $s$ determines the topology of the geometry at infinity, namely, $S^3/\mathbb{Z}_s$ .

- 微扰层面，和四维情况一样，量子修正仅出现在单圈。显式计算或几何论证 [28,29] 表明，(47) 式中的场 $w$ 本质上就是狄拉克磁单极。若 $\varphi = (\phi, \theta, \varphi)$ ，则 $w_i d\varphi^i = \; s\cos\theta d\phi/2$ ，其中整数 $s$ 决定了无穷远处几何的拓扑，即 $S^3/\mathbb{Z}_s$ 。

- There can also be instanton effects (the monopole in four dimensions become an instanton in three dimensions). The counting of zero modes determine these quantum corrections: in the U(1) gauge theory, there is no such correction, while the SU(2) gauge theory with $N_f \leq 2$ has such corrections. $\mathcal{N} = 4$ supersymmetry in three dimensions gives zero modes contributing to a fermionic vertex $\psi\psi\psi\psi$ which is the supersymmetric completion of a correction to the metric.

- 还可能存在瞬子效应 (四维中的单极子会成为三维中的瞬子)。零模计数决定了这些量子修正: 在 U(1) 规范理论中，不存在这类修正，而带有 $N_f \leq 2$ 的 SU(2) 规范理论则存在这类修正。三维中的 $\mathcal{N} = 4$ 超对称会给出贡献费米子顶点 $\psi\psi\psi\psi$ 的零模，该顶点是度规修正的超对称完备化。

## U(1) Gauge Theory with $N_f$ Hypermultiplets of Charge 1

## 带 $N_f$ 个电荷 1 超多重子的 U(1) 规范理论

Let us consider the U(1) gauge theory with $N_f$ hypermultiplets of charge 1 . The correction is one-loop exact and there is no instanton correction. The fact that the Coulomb branch moduli space is hyperKähler determines the metric completely

我们来考虑带 $N_f$ 个电荷 1 超多重子的 U(1) 规范理论。该修正为单圈精确，不存在瞬子修正。库仑分支模空间是超凯勒流形这一条件完全确定了度规

$$ds^2 = g^{-2}(\mathbf{x})(d\mathbf{x})^2 + g^2(\mathbf{x})(da + \mathbf{w}\cdot d\mathbf{x})^2 \tag{48}$$

given the function $g(\mathbf{x})$ , via $\nabla g^{-2} = \nabla\times\mathbf{w}$ . We can easily calculate the correction to $g^{-2}(\mathbf{x})$ at 1-loop, by considering the contributions of the hypermultiplets of mass $|\mathbf{x}-\mathbf{m}_i|$ to the two-point function of the vector multiplet fields. The relevant Feynman graph is a loop of massive particles

给定函数 $g(\mathbf{x})$ ，通过 $\nabla g^{-2} = \nabla\times\mathbf{w}$ 得到。我们可以很容易地计算 $g^{-2}(\mathbf{x})$ 的单圈修正，方法是考虑质量为 $|\mathbf{x}-\mathbf{m}_i|$ 的超多重子对向量多重子场两点函数的贡献。相关费曼图是一个有质量粒子的圈

$$\int \frac{d^3p}{(p^2+m^2)^2} \sim \frac{1}{m} \tag{49}$$

which gives

其结果为

$$g^{-2}(\mathbf{x}) = g_{\mathrm{cl}}^{-2} + \sum_{i=1}^{N_f} \frac{1}{|\mathbf{x} - \mathbf{m}_i|}. \tag{50}$$

For large $x$ , we have $g^{-2}(\mathbf{x}) \sim$ const. $+N_f/|\mathbf{x}|$ . The integer $N_f$ will appear in $\mathbf{w}$ and determines the topology at infinity, namely, $S^3/\mathbb{Z}_{N_f}$ . This is the well-known Taub-NUT metric [30, 31] .

对于大 $x$ ，我们有 $g^{-2}(\mathbf{x}) \sim$ 常数$\cdot+N_f/|\mathbf{x}|$ 。整数 $N_f$ 会出现在 $\mathbf{w}$ 中，决定了无穷远的拓扑，即 $S^3/\mathbb{Z}_{N_f}$ 。这就是著名的陶布-纽特 (Taub-NUT) 度规 [30, 31] 。

The same result can also be obtained from the system of D2- and D6-branes in type IIA. We will see that the quantum effects in the field theory discussed above arise purely from the classical geometry of space-time. We consider the D2-brane as a probe of the type IIA geometry deformed by the presence of the D6-branes. The D2-brane coupling is given by the dilaton at the point $\mathbf{x}$ in the directions (789) where the D2-brane is located. The D6-branes are point sources of the dilaton in the (789) directions (recall that the position of the D6-branes in the (789) directions is the bare mass of the hypermultiplet) and so

相同结果也可以从 IIA 型弦论中 D2 膜与 D6 膜的系统得到。我们会看到，上文讨论的场论量子效应完全来源于时空的经典几何。我们将 D2 膜视作被 D6 膜变形的 IIA 几何的探针。D2 膜的耦合由 D2 膜所在位置 (789) 方向 $\mathbf{x}$ 点的胀子给出。D6 膜是 (789) 方向上胀子的点源 (注意 D6 膜在 (789) 方向的位置对应超多重子的裸质量)，因此

$$\Box e^{-\phi} = \sum_{i=1}^{N_f} \delta^{(3)}(\mathbf{x} - \mathbf{m}_i). \tag{51}$$

This is the Laplace equation in three dimensions whose solution is

这是三维空间的拉普拉斯方程，其解为

$$e^{-\phi} = e^{-\phi_0} + \sum_{i=1}^{N_f} \frac{1}{|\mathbf{x} - \mathbf{m}_i|}. \tag{52}$$

Identifying $g^{-2}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$ , we obtain (50), as required.

通过对 $g^{-2}(\mathbf{x}) = e^{-\phi(\mathbf{x})}$ 做等同化，我们得到要求的式 (50)。

The fact that the gauge field can be dualized to a real scalar $a$ suggests that there is a new space-time direction opening up. This is consistent with the fact that M-theory can be defined as a strong coupling limit of type IIA string theory. More precisely, type IIA string theory is equivalent to M-theory on $\mathbb{R}^{1,9} \times S^1$ where the radius of the circle increases with the string coupling. The compact scalar $a$ , which we can redefine as $x^{10}$ , parametrizes the motion along $S^1$ . This is consistent with the fact that the radius of $a$ increases with the gauge coupling $g$ (which is determined by the string coupling).

规范场可以对偶化为实标量 $a$ 这一事实暗示存在一个新的时空方向开启。这与 M 理论可定义为 IIA 型弦论强耦合极限的性质一致。更准确地说，IIA 型弦论等价于 $\mathbb{R}^{1,9}\times S^1$ 上的 M 理论，其中圆的半径随弦耦合常数增大。我们可以重新将紧致标量 $a$ 定义为 $x^{10}$ ，它参数化了沿 $S^1$ 方向的运动。这与 $a$ 的半径随规范耦合常数 $g$ (由弦耦合常数决定) 增大的性质一致。

It is interesting to uplift our system of D2- and D6-branes to M-theory. Let us analyze the brane content of type IIA and compare it to that of M-theory. The type IIA theory contains D0-, D2-, D4-, NS5- (charged under $B_6$ dual to $B_2$ ), D6-, and D8-branes. On the other hand, M-theory contains only M2- and M5-branes, charged under the M-theory three-form $A^{(3)}$ and its dual $A^{(6)}$ , respectively. How can we obtain the type IIA branes from the M-theory ones by compactifying along $x^{10}$ ? For convenience, let us tabulate some answers to this question. (We shall not discuss here the D8-brane, which is charged under the non-dynamical $C_9$ field and introduces a cosmological constant. It becomes important to describe massive type IIA string theory.)

将我们的 D2-D6 膜系统上升级到 M 理论是很有意思的。我们来分析 IIA 弦论的膜内容，并和 M 理论对比。IIA 理论包含 D0 膜、D2 膜、D4 膜、NS5 膜 (带电于对偶于 $B_2$ 的 $B_6$ )、D6 膜和 D8 膜。另一方面，M 理论仅包含 M2 膜和 M5 膜，分别带电于 M 理论三形式场 $A^{(3)}$ 及其对偶 $A^{(6)}$ 。我们如何通过沿 $x^{10}$ 的紧化从 M 理论膜得到 IIA 理论膜？为方便起见，我们将这个问题的部分结论列成表格。(此处我们不讨论 D8 膜，它带电于非动力学的 $C_9$ 场，会引入宇宙学常数，它在描述有质量 IIA 弦论时很重要。)

| M-theory | Charged under | Type IIA | Charged under |
|---|---|---|---|
| M2 | $A^{(3)}_{\mu\nu\rho}$ | D2 | $C_3$ |
| M2 wrapped $x^{10}$ | $A^{(3)}_{\mu\nu 10}$ | F1 (fundamental string) | $B_2$ |
| M5 | $A^{(6)}$ | NS5 | $B_6$ |
| M5 wrapped $x^{10}$ | $A^{(6)}_{\mu_1\cdots\mu_9 10}$ | D4 | $C_5$ |

(53)

In M-theory, the D2-brane becomes an M2-brane that, in fact, has eight real scalar fields on its worldvolume, namely, $\mathbf{x}=(x_7,x_8,x_9)$ , $x_{10}$ , and the four scalars associated with the directions (3456) and corresponding to the adjoint decoupled hypermultiplet.

在 M 理论中,D2 膜会变成 M2 膜,实际上 M2 膜的世界体上存在八个实标量场,即 $\mathbf{x}=(x_7,x_8,x_9)$ , $x_{10}$ ，其中四个标量对应 (3456) 方向，对应退耦合的伴随超多重子。

What happens to the D0- and D6-branes? The D0-brane, which is charged under $(C_1)_\mu = g_{\mu 10}$ , is the Kaluza-Klein mode of the metric. The D6-brane is charged under the dual of $C_1$ and should be the Kaluza-Klein (KK) monopole. These objects have been well-studied: a metric with a U(1) isometry (associated with a compact coordinate $a$ ) after compactification produces a soliton magnetically charged under the gauge field $g_{\mu 10}$ . And indeed, the Taub-NUT is the KK monopole par excellence. Therefore, the D6-brane, upon uplifting to M-theory, does not become a soliton charged under some antisymmetric field, but it rather corresponds to a background with a nontrivial metric.

那么 D0 膜和 D6 膜会发生什么呢？带 $(C_1)_\mu = g_{\mu 10}$ 荷的 D0 膜是度规的卡鲁扎-克莱因模式。D6 膜带 $C_1$ 对偶的荷，应当是卡鲁扎-克莱因 (KK) 磁单极。这些对象已经得到了充分研究: 具有 U(1) 等距 (对应紧致坐标 $a$ ) 的度规紧致化后，会产生一个带规范场 $g_{\mu 10}$ 磁荷的孤子。事实上，陶布-纽特 (Taub-NUT) 空间就是典型的 KK 磁单极。因此，D6 膜提升到 M 理论后，并不会变成某种反对称场下带电的孤子，而是对应一个具有非平凡度规的背景。

The system of a D2-brane and $N_f$ D6-branes, upon uplifting to M-theory, becomes an M2-brane on the background with an $N_f$ -centered Taub-NUT metric in the direction (78910). We expect that the action for the eight scalars of the M2- brane is

一个 D2 膜与 $N_f$ 个 D6 膜构成的系统，提升到 M 理论后，就变成了 (78910) 方向上带有以 $N_f$ 为中心的陶布-纽特度规的背景中的 M2 膜。我们预期，M2 膜的八个标量满足的作用量为

$$g_{ij} dx^i dx^j + g_{ab}^{\mathrm{TN}} dx^a dx^b, \tag{54}$$

where $i, j = 3, 4, 5, 6$ and $a, b = 7, 8, 9, 10$ . The former part decouples and becomes free. We see that this is indeed the moduli space obtained from the field theory. The M2-brane behaves again as a probe brane.

其中 $i, j = 3, 4, 5, 6$ 和 $a, b = 7, 8, 9, 10$ 。前一部分退耦合后变为自由项。我们可以看到，这确实是从场论得到的模空间，M2 膜在这里仍表现为探针膜。

If $g_{\mathrm{cl}} \to \infty$ (i.e., the theory is on $\mathbb{R}^{11}$ ), the constant term in $g^{-2}(\mathbf{x})$ disappears, and the metric becomes that of an ALE space. As we have already discussed, the ALE spaces are four-dimensional hyperKähler manifolds that can be obtained as a resolution of the algebraic singularity $\mathbb{C}^2/\Gamma$ , where $\Gamma$ is a discrete subgroup of SU(2) classified by the simply laced ADE algebras. In a smooth ALE space, the singularity is replaced by two spheres with an intersection matrix corresponding to the affine Dynkin diagram of the corresponding ADE group. In the case of the D6-branes, we have the ALE space associated with the resolution of $\mathbb{C}^2/\mathbb{Z}_{N_f}$ . From this point of view, the gauge fields on the D6-branes arise as the reduction of the M-theory three-form $A_3$ on the $N_f$ two spheres and the positions of the D6-branes $\mathbf{m}_i$ as the blow-up parameters. For $\mathbf{m}_i$ different from each other, the D6- branes are separated, and the ALE space is smooth. When all the $\mathbf{m}_i$ are coincident, the D6-branes are on top of each other, and the ALE space becomes the singular orbifold $\mathbb{C}^2/\mathbb{Z}_{N_f}$ . At this particular point of the moduli space, the two spheres with intersection matrix corresponding to the Dynkin diagram of the ADE groups $\mathrm{U}\left(N_f\right)$ become of zero volume. M2-branes wrapping these vanishing two cycles give extra massless fields and become the non-abelian gauge bosons that enhance the D6-brane gauge group $\mathrm{U}(1)^{N_f}$ to $\mathrm{U}\left(N_f\right)$ .

如果 $g_{cl} \to \infty$ (即理论定义在 $\mathbb{R}^{11}$ 上)，$g^{-2}(\mathbf{x})$ 中的常数项会消失，度规就成为 ALE 空间的度规。正如我们之前讨论过的，ALE 空间是四维超凯勒流形，可以通过对代数奇点 $\mathbb{C}^2/\Gamma$ 求解得到，其中 $\Gamma$ 是 SU(2) 的离散子群，由单系 ADE 代数分类。在光滑 ALE 空间中，奇点被一系列二维球面取代，这些球面的相交矩阵对应相关 ADE 群的仿射 Dynkin 图。对于 D6 膜的情况，我们得到的 ALE 空间对应 $\mathbb{C}^2/\mathbb{Z}_{N_f}$ 奇点的解。从这个角度看，D6 膜上的规范场来源于 M 理论三形式场 $A_3$ 在 $N_f$ 个二维球面上的约化，而 D6 膜的位置 $\mathbf{m}_i$ 就是爆破参数。当所有 $\mathbf{m}_i$ 互不相等时，D6 膜相互分离，ALE 空间是光滑的。当所有 $\mathbf{m}_i$ 重合时，D6 膜堆叠在一起，ALE 空间就成为奇异轨形 $\mathbb{C}^2/\mathbb{Z}_{N_f}$ 。在模空间的这个特殊点上，对应 ADE 群 $\mathrm{U}(N_f)$ Dynkin 图的二维球面体积变为零。M2 膜包裹这些零体积闭圈会产生额外的零质量场，这些零质量场就是非阿贝尔规范玻色子，会将 D6 膜的规范群 $\mathrm{U}(1)^{N_f}$ 增强为 $\mathrm{U}(N_f)$ 。

## SU(2) Gauge Theory with $N_f$ Hypermultiplets in the Fundamental Representation

## 基础表示中含 $N_f$ 超多重子的 SU(2) 规范理论

As we already discussed, the classical metric is $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ . For $N_f > 2$ , there is a one-loop perturbative correction but no instanton correction. The perturbative correction leads to a Taub-Nut geometry of type $D$ . The metric at infinity is determined by the function (the negative term -4 is the contribution of the gauge group $\mathrm{SU}(2)$ , which is in fact $-2N_c$ for general $\mathrm{SU}(N_c)$ )

如我们此前讨论，经典度规为 $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ 。对于 $N_f > 2$ ，存在单圈微扰修正，但无瞬子修正。微扰修正给出 $D$ 型 Taub-Nut 几何。无穷远处的度规由下述函数确定 (负项-4 是规范群 $\mathrm{SU}(2)$ 的贡献，对一般 $\mathrm{SU}(N_c)$ 而言该贡献实际为 $-2N_c$ )

$$g^{-2}(\mathbf{x}) \sim \text{ const. } + \frac{2N_f - 4}{|\mathbf{x}|} \tag{55}$$

that reconstructs a $S^3/\mathbb{D}_{N_f-2}$ bundle [29], where $\mathbb{D}_{N_f-2}$ is the dihedral group of order $2N_f-4$ . For $N_f = 2$ , there are no quantum corrections, and the quantum metric is precisely the flat metric on $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ . For $N_f = 0, 1$ , there are both 1-loop and instanton corrections.

该函数重构了 $S^3/\mathbb{D}_{N_f-2}$ 丛 [29]，其中 $\mathbb{D}_{N_f-2}$ 是阶为 $2N_f - 4$ 的二面体群。对于 $N_f = 2$ ，不存在量子修正，量子度规恰好是 $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ 上的平坦度规。对于 $N_f = 0, 1$ ，同时存在 1 圈修正和瞬子修正。

To understand what the fully quantum corrected metric should be, let us consider this from the string theory perspective. Consider first the case of zero masses corresponding to a set of $N_f$ D6-branes on top of the orientifold plane. For $N_f \geq 2$ , we expect the gauge symmetry on the D6-branes to be non-abelian and a singular geometry in M-theory, a Taub-Nut-like geometry of type $D$ which becomes the ALE singularity $\mathbb{C}^2/\mathbb{D}_{N_f-2}$ at infinite coupling. However, for $N_f = 0$ and 1, we expect the space to be completely smooth.

为了得到全量子修正后的正确度规，我们从弦论视角来分析这一问题。首先考虑零质量情形，对应奥连提福德平面上堆叠的 $N_f$ 张 D6 膜。对于 $N_f \geq 2$ ，我们预期 D6 膜上的规范对称性为非阿贝尔群，M 理论中几何奇异，是 $D$ 型 Taub-Nut 类几何，在无穷耦合下退化为 ALE 奇点 $\mathbb{C}^2/\mathbb{D}_{N_f-2}$ 。但对于 $N_f = 0$ 和 1 的情况，我们预期空间是完全光滑的。

Consider the $N_f = 0$ and 1 cases. The one-loop contribution (55) to the metric becomes negative for $N_f = 0$ and 1, and it is modified by instanton corrections. The fact that the metric has to be smooth hyper-Kähler with symmetry $\mathrm{SU}(2)_V$ restricts to it to only two possibilities: (1) the Atiyah-Hitchin manifold $\mathcal{N}$ , which is isomorphic to the moduli space of two monopoles and (2) its simply connected double cover $\overline{\mathcal{N}}$ . By considering the topology of the metric at infinity, these can be identified with the moduli spaces for $N_f = 0$ and $N_f = 1$ , respectively.

我们来看 $N_f = 0$ 和 1 的情况。度规的单圈贡献 (55) 在 $N_f = 0$ 和 1 时为负，因此需要瞬子修正修正该项。度规必须是带对称性 $\mathrm{SU}(2)_V$ 的光滑超凯勒流形，这一要求将结果限制为仅两种可能:(1) 阿蒂亚-希钦流形 $\mathcal{N}$ ，它同构于两个单极子的模空间；(2) 它的单连通双叶覆盖 $\overline{\mathcal{N}}$ 。通过分析无穷远处度规的拓扑，可分别确定这两个流形对应 $N_f = 0$ 和 $N_f = 1$ 的模空间。

Adding masses desingularizes these spaces. For $N_f \geq 2$ , these are the blow-up parameters of the singularity of type D. For $N_f = 1$ , one finds a metric that depends on three parameters $\mathbf{m}$ in $\mathrm{SU}(2)_V$ . The metric of this type was studied by Dancer [32].

引入质量会使这些空间退奇异。对于 $N_f \geq 2$ ，质量就是 D 型奇点的吹胀参数。对于 $N_f = 1$ ，可得到依赖于 $\mathrm{SU}(2)_V$ 中三个参数 $\mathbf{m}$ 的度规。该类型度规已由 Dancer 研究 [32]。

The $N_f = 2$ case is rather peculiar. As we have discussed, there is no 1-loop correction. In fact, there is also no instanton correction and so the metric is classically exact, namely, $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ if all of the masses are zero. This can be understood from string theory as follows: If the masses $\mathbf{m}_1$ and $\mathbf{m}_2$ are zero, then both D6-branes are on the $\mathrm{O}_6^-$ orientifold plane. Recall that the $\mathrm{O}_6^-$ plane has charge -2 . This is precisely cancelled (locally) by the charge of two D6-branes, each of which carries charge +1 . Hence, there is no source for the dilaton. However, if $\mathbf{m}_1 \neq \mathbf{m}_2 \neq \mathbf{0}$ , there are one object of charge -2 located at0and two objects of charge +1 located at $\mathbf{m}_1$ and $\mathbf{m}_2$ , and there is a need for a nontrivial dilaton.

$N_f = 2$ 的情况相当特殊。正如我们之前讨论的，这里不存在 1 圈修正。事实上，也没有瞬子修正，因此度规是经典精确的，即当所有质量都为零时度规为 $\left(\mathbb{R}^3 \times S^1\right)/\mathbb{Z}_2$ 。这可以从弦论角度做如下理解: 如果质量 $\mathbf{m}_1$ 和 $\mathbf{m}_2$ 都为零，那么两张 D6 膜都位于 $\mathrm{O}_6^-$ 定向偶面上。我们知道 $\mathrm{O}_6^-$ 面带电量为-2，这正好被两张各带 +1 电量的 D6 膜 (局部) 抵消了。因此，这里没有 dilaton 的源。但如果是 $\mathbf{m}_1 \neq \mathbf{m}_2 \neq \mathbf{0}$ 的情况，就会有一个带-2 电量的物体位于 0 点，两个带 +1 电量的物体分别位于 $\mathbf{m}_1$ 和 $\mathbf{m}_2$ ，此时 dilaton 就必须非平凡。

We have mentioned that the metric of the moduli space of SU(2) gauge theory with $N_f = 0, 1$ is the metric for the moduli space of two SU(2) monopoles. In section ”Hanany-Witten Brane Configurations in Three Dimensions,” we will explain the reason for this and generalize this to the moduli space of $N\mathrm{SU}(2)$ monopoles.

我们已经提到，带有 $N_f = 0, 1$ 的 SU(2) 规范理论模空间的度规，就是两个 SU(2) 磁单极子模空间的度规。在“三维中的 Hanany-Witten 膜构型”一节，我们会解释其中的原因，并将其推广到 $N$SU (2) 个磁单极子的模空间。

## The Quantum Coulomb Branch

## 量子库仑分支

In general, the quantum Coulomb branch has an intricate mathematical structure and no simple characterization such as the hyperKähler quotient description (34) valid for the Higgs branch. A proper mathematical characterization of the quantum corrected Coulomb branch has been proposed by Nakajima and collaborators [33, 34]. On a related physical side, some information can be extracted from the spectrum of BPS operators, corresponding to the set of holomorphic functions on the moduli space whose partition function is encoded in the monopole formula [35] and from their OPE [36]. The chiral ring of BPS operators associated with the Coulomb branch has a complicated structure involving monopole operators in addition to the classical fields in the Lagrangian [37, 38] .

一般而言，量子库仑分支拥有复杂的数学结构，不存在希格斯分支所具备的超卡勒商描述 (34) 这类简单表征方法。中岛幸雄及其合作者已经提出了对量子修正库仑分支的恰当数学表征 [33, 34]。在相关物理层面，我们可以从 BPS 算符的能谱中提取部分信息——BPS 算符对应模空间上的全纯函数集合，其配分函数由单极公式 [35] 刻画，也可以从这些算符的算符乘积展开中获取信息 [36]。与库仑分支关联的 BPS 算符手征环结构十分复杂，除拉格朗日量中的经典场外，还包含单极算符 [37, 38] 。

## Hanany-Witten Brane Configurations in Three Dimensions

## 三维中的汉纳尼-威滕膜构型

Let us consider a D3-brane stretched in the (0126) directions and two NS5-branes stretched in the (012345) directions. Note that all of the branes become points from the perspective of an observer in the (789) directions.

我们来考虑一张沿 (0126) 方向延伸的 D3 膜，以及两张沿 (012345) 方向延伸的 NS5 膜。注意，从 (789) 方向观测者的视角来看，所有膜都表现为点。

As we have discussed, the D3-brane can end on a D5-brane. The former turns on the field $F_{\mu\nu}$ on the D5-brane. Using $S$ -duality, which exchanges the D5- and NS5-branes and maps a D3-brane to itself, we see that the above brane configuration is valid and that the field $F_{\mu\nu}$ on the NS5-brane must be turned on.

正如我们此前讨论过的，D3 膜可以终止在 D5 膜上。D3 膜会在 D5 膜上激发场 $F_{\mu\nu}$ 。利用交换 D5 膜与 NS5 膜、且将 D3 膜映射为自身的 $S$ 对偶，我们可知上述膜构型是成立的，且必须在 NS5 膜上激发场 $F_{\mu\nu}$ 。

The theory on the D3-brane is three-dimensional, corresponding to the directions (012). Indeed, the direction (6) is finite and brings about only KK modes. The gauge coupling g is given by

D3 膜上的理论是三维的，对应 (012) 方向。事实上，(6) 方向是有限的，仅带来 KK 模。规范耦合 g 由下式给出

$$\frac{1}{g^2} = Le^{-\phi} \tag{56}$$

where $L$ is the distance between the two NS5-branes in the (6) direction. The field content depends on the condition that is imposed at the boundary of the D3-brane. Since the theory on the D3-branes by itself has 16 supercharges, we expect that the presence of the NS5-branes breaks supersymmetry to eight supercharges. In terms of 3 d$\mathcal{N} = 4$ supersymmetry, the vector multiplet on the D3-brane decomposes into a vector multiplet $(A_\mu, \mathbf{x})$ where $\mathbf{x}$ is the triplets of scalars parametrizing the (345) directions and the hypermultiplet $(A_6, \mathbf{y})$ where $\mathbf{y}$ are the scalars in the (789) directions. The D3-brane cannot oscillate in the (789) directions because the NS5- brane is larger in dimension, is more rigid, and is fixed at a particular point. We thus expect that the hypermultiplets get projected away by the boundary condition and that only the vector multiplet survives.

其中 $L$ 是两张 NS5 膜在 (6) 方向上的距离。场的内容取决于 D3 膜边界上施加的边界条件。由于 D3 膜本身的理论具有 16 个超荷，我们预期 NS5 膜的存在会将超对称破缺为 8 个超荷。在 3 d$\mathcal{N} = 4$ 超对称的框架下，D3 膜上的矢量多重态分解为一个矢量多重态 $(A_\mu, \mathbf{x})$，其中 $\mathbf{x}$ 是参数化 (345) 方向的三个标量，以及一个超多重态 $(A_6, \mathbf{y})$，其中 $\mathbf{y}$ 是对应 (789) 方向的标量。D3 膜无法在 (789) 方向振荡，因为 NS5 膜维度更高、更刚性，且固定在特定点上。因此我们预期超多重态会被边界条件投影掉，仅矢量多重态留存下来。

Let us discuss more explicitly the supersymmetry preserved by this brane configuration. For D3- and NS5-branes, we have

我们来更明确地讨论该膜构型保留的超对称。对于 D3 膜和 NS5 膜，我们有

$$\mathrm{D3} : \varepsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_6\varepsilon_R$$

(57)

$$\mathrm{NS5} : \varepsilon_L = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\varepsilon_L,\ \varepsilon_R = -\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\varepsilon_R\ .$$

The condition for the NS5-brane comes from the fact that the NS5-brane is a closed string object, and so the conditions for $\varepsilon_L$ and $\varepsilon_R$ are independent. Combining the two conditions, we have

NS5 膜的条件来自于 NS5 膜是闭弦客体这一事实，因此 $\varepsilon_L$ 和 $\varepsilon_R$ 的条件是独立的。结合两个条件，我们得到

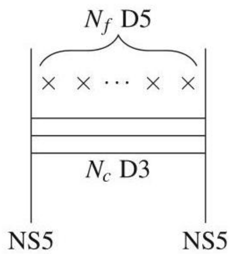
$$\varepsilon_L = \Gamma_3\Gamma_4\Gamma_5\Gamma_6\varepsilon_R = \Gamma_0\Gamma_1\Gamma_2\Gamma_7\Gamma_8\Gamma_9\varepsilon_R, \tag{58}$$

where in the second equality we have used that type IIB spinors are chiral, $\Gamma_{11}\varepsilon_R = \Gamma_0\Gamma_1\cdots\Gamma_9\varepsilon_R = \varepsilon_R$ . Indeed, this is clearly compatible with the fact that the amount of supersymmetry is broken by 1/4 due to the brane system. Observe also that the second equality is precisely the condition for a D5-brane spanning the direction (012789). In other words, inserting a D5-brane in the (012789) direction does not break further supersymmetry.

其中第二个等式我们用到了 IIB 型弦的旋量是手性的这一性质, $\Gamma_{11}\varepsilon_R = \Gamma_0\Gamma_1\cdots\Gamma_9\varepsilon_R = \varepsilon_R$。这显然符合该膜系统将超对称破缺为原来的 1/4 这一结论。还可以注意到，第二个等式恰好是覆盖 (012789) 方向的 D5 膜满足的条件。换句话说，在 (012789) 方向插入一张 D5 膜不会进一步破缺超对称。

Let us then add also D5-branes. The number of the DN coordinates between D3- and D5-branes is four, and so the 3-5 open strings produce hypermultiplets in the fundamental representation of the gauge group on the D3-brane. Let us denote by $\mathbf{m}$ the position of the D5-branes in (345) directions. The mass of the hypermultiplets is then $\mathbf{m}-\mathbf{x}$ , where $\mathbf{x}$ is the position of the D3-branes inside the NS-brane in the (345) directions (associated with the three scalars in the vector multiplet of the gauge group), as expected from the field theory. In this way, with $N_c$ D3-branes, $N_f$ D5-branes, and two NS5-branes (see (59)), we can realize the $\mathrm{U}(N_c)$ gauge theory with $N_f$ hypermultiplets in the fundamental representations from the brane system:

接下来我们再加入 D5 膜。D3 膜与 D5 膜之间的 DN 坐标数目为 4，因此 3-5 开弦会在 D3 膜的规范群下生成基础表示的超多重态。我们用 $\mathbf{m}$ 表示 D5 膜在 (345) 方向的位置。那么超多重态的质量为 $\mathbf{m}-\mathbf{x}$，其中 $\mathbf{x}$ 是 D3 膜在 NS 膜内 (345) 方向的位置 (对应规范群矢量多重态中的三个标量)，这和场论的预期一致。通过这种方式，若拥有 $N_c$ 张 D3 膜、$N_f$ 张 D5 膜和两张 NS5 膜 (见式 (59))，我们可以从该膜系统实现具有 $N_f$ 个基础表示超多重态的 $\mathrm{U}(N_c)$ 规范理论:




(59)

We will refer to this type of brane setup as the Hanany-Witten brane configuration or construction [1]. Note that one difference between this method and the D2-D6- brane system is that there is no adjoint hypermultiplet in the former, whereas there is one in the latter.

我们将这类膜结构称为汉纳尼-威滕膜构造或汉纳尼-威滕膜结构 [1]。注意，该方法与 D2-D6 膜系统的一个区别在于，前者不存在伴随超多重子，而后者存在一个伴随超多重子。

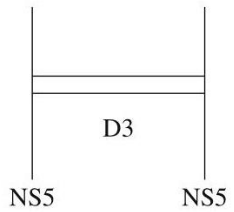
## Application 1: Monopole Moduli Space

## 应用 1: 磁单极模空间

The first application is to understand why the Coulomb branch of the $3d\mathcal{N}=4$ SU(2) (resp. $\mathrm{SU}(N)$ ) pure SYM is identical to the moduli space of two (resp. $N$ ) SU(2) monopoles. The U(2) pure SYM can be realized on two D3-branes stretched between two NS5-branes:

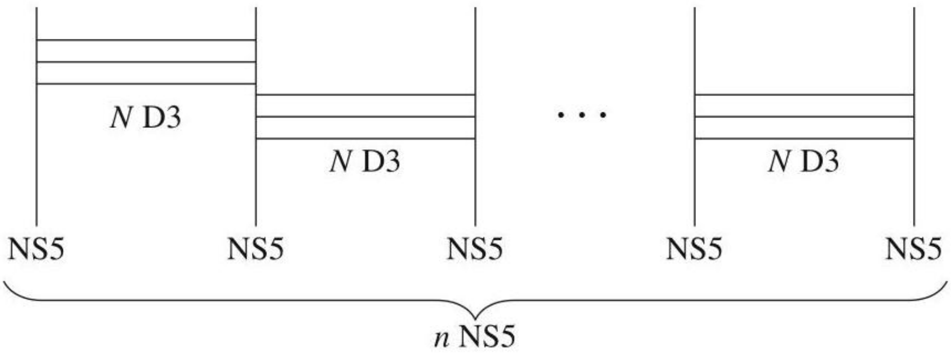
第一个应用是解释为何三维 $3d\mathcal{N}=4$ SU(2)(对应 SU $(N)$ ) 纯超对称杨-米尔斯理论的库仑分支，恰好等同于两个 (对应 $N$ )SU(2) 磁单极的模空间。U(2) 纯超对称杨-米尔斯理论可以通过张在两根 NS5 膜之间的两根 D3 膜实现:

(60)




In this case, the U(1) factor decouples, and so we are left with the SU(2) gauge group. This brane system has a global symmetry coming from the presence of the non-dynamical brane of higher dimension. The theory on two NS5-branes is the six-dimensional $\mathcal{N}=1$ theory with group SU(2). The latter is spontaneously broken to U (1) due to the separation of the branes in the (6) direction, where the distance $\Delta x^6$ between them corresponds to the VEV of the Higgs field. Moreover, the vacuum equation tells us that the D3-brane is a monopole for the field living on the NS5- brane. One then understands why the Coulomb branch of the 3d theory coincides with the moduli space of two SU(2) monopoles. The distance between two D3- branes in the (456) directions, namely, $\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$ , is the distance between the two monopoles. (The free parameter in the Atiyah-Hitchin metric which is also the VEV $\Delta x^6$ of the Higgs field is identified with the gauge coupling of the D3- brane theory.) This result can be easily generalized. For example, we can consider the 3 d$\mathcal{N}=4$ pure U $(N)$ SYM, which is realized on the $N$ D3-branes stretched between two NS5-branes. Its Coulomb branch is identified with the moduli space of $N$SU (2) monopoles. We can also generalize this further to the moduli space of $N$SU $(n)$ monopoles by considering the system of $n$NS 5-branes with $N$D 3-branes stretched between each pair of them that are next to each other:

在这个情形下，U(1) 因子退耦，我们得到 SU(2) 规范群。这个膜系统由于存在更高维的非动力学膜而具有整体对称性。两张 NS5 膜上的理论是六维 $\mathcal{N}=1$ 理论，规范群为 SU(2)。由于膜在 (6) 方向分离，该规范群自发破缺为 U (1) ，膜之间的距离 $\Delta x^6$ 对应希格斯场的真空期望值。此外，真空方程表明，D3 膜是 NS5 膜上生存场的磁单极。由此我们可以理解为何三维理论的库仑分支与两个 SU(2) 磁单极的模空间一致。两张 D3 膜在 (456) 方向的距离，也就是 $\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2$ ，就是两个磁单极之间的距离。(Atiyah-Hitchin 度量中的自由参数同时也是希格斯场的真空期望值 $\Delta x^6$ ，对应 D3 膜理论的规范耦合。) 这个结论可以很容易推广。例如，我们可以考虑 3 d$\mathcal{N}=4$ 纯 U $(N)$ 超对称杨-米尔斯理论，它由张在两张 NS5 膜之间的 $N$ 张 D3 膜实现，其库仑分支对应 $N$SU (2) 个磁单极的模空间。我们还可以进一步推广，得到 $N$SU $(n)$ 个磁单极的模空间: 考虑由 $n$NS 张 5 膜构成的系统，相邻每对 5 膜之间都张有 $N$D 张 3 膜:




(61)

We will see in the next subsection that this realizes the 3 d$\mathcal{N} = 4$ theory with $\mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \cdots \times \mathrm{U}(N)_{n-1}$ gauge group with hypermultiplets in the $\left(\mathbf{N}, \overline{\mathbf{N}}\right)$ representation of $\mathrm{U}(N)_i \times \mathrm{U}(N)_{i+1}$ with $i = 1, 2, \ldots, n-2$ .

我们将在下一小节看到，这个系统实现了 3 d$\mathcal{N} = 4$ 理论，它具有 $\mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \cdots \times \mathrm{U}(N)_{n-1}$ 规范群，且在 $\mathrm{U}(N)_i \times \mathrm{U}(N)_{i+1}$ 的 $\left(\mathbf{N}, \overline{\mathbf{N}}\right)$ 表示中含有超多重子，满足 $i = 1, 2, \ldots, n-2$ 。

## General Rules of the Game

## 游戏的一般规则

We can consider a system of D3-branes stretched between two D5-branes. $S$ -duality can help us understand what is going on. On the D3-brane, $S$ -duality is the electric-magnetic duality which transforms $F_{\mu\nu} \leftrightarrow \varepsilon_{\mu\upsilon\tau\rho} F^{\tau\rho}$ where $\mu = 0, 1, 2, 6$ . In particular, the scalar $A_6$ gets exchanged with the scalar $a$ dual to the gauge field in three dimensions: $\partial_\mu A_6 = F_{\mu 6} \leftrightarrow \varepsilon_{\mu 6\tau\sigma} F^{\tau\rho} \equiv \partial_\mu a$ implies $A_6 \leftrightarrow a$ . Therefore, the vector multiplet $(a, \mathbf{x})$ gets exchanged with the hypermultiplet $(A_6, \mathbf{y})$ , where, as before, $\mathbf{x}$ is a vector in the (345) directions and $\mathbf{y}$ is a vector in the (789) directions.

我们可以研究一组伸展在两张 D5 膜之间的 D3 膜系统。$S$ 对偶可以帮助我们理解其中的物理过程。在 D3 膜上，$S$ 对偶就是电-磁对偶，它变换 $F_{\mu\nu} \leftrightarrow \varepsilon_{\mu\upsilon\tau\rho} F^{\tau\rho}$ ，其中 $\mu = 0, 1, 2, 6$ 。特别地，标量场 $A_6$ 会与三维中对偶于规范场的标量场 $a$ 交换: $\partial_\mu A_6 = F_{\mu 6} \leftrightarrow \varepsilon_{\mu 6\tau\sigma} F^{\tau\rho} \equiv \partial_\mu a$ 意味着 $A_6 \leftrightarrow a$ 。因此，矢量多重态 $(a, \mathbf{x})$ 会与超多重态 $(A_6, \mathbf{y})$ 交换，其中和之前一样，$\mathbf{x}$ 是 (345) 方向的矢量，$\mathbf{y}$ 是 (789) 方向的矢量。

As a consequence, the boundary conditions also get exchanged. For a D3- brane stretched between two D5-branes, the hypermultiplet $(A_6, \mathbf{y})$ survives the projection. This is because the positions $\mathbf{x}$ in the (345) directions are fixed (due to the fact that the D5-brane is big and rigid), and so the D3-brane can only oscillate within the D5-brane with the positions $\mathbf{y}$ in the (789) direction. Under $S$ - duality, the theory should not be changed. However, apparently, a vector multiplet is transformed into a hypermultiplet. This means that the duality makes the vector multiplet indistinguishable from the hypermultiplet in three dimensions.

因此，边界条件也会发生交换。对于伸展在两张 D5 膜之间的 D3 膜，超多重态 $(A_6, \mathbf{y})$ 在投影后保留下来。这是因为 (345) 方向的位置 $\mathbf{x}$ 是固定的 (由于 D5 膜体积大且刚性)，因此 D3 膜只能沿着 D5 膜在 (789) 方向以位置 $\mathbf{y}$ 振荡。在 $S$ 对偶下，理论本身不发生改变。但表面上看，矢量多重态被变换为超多重态。这说明，该对偶使得矢量多重态在三维中与超多重态不可区分。

We remark that it is a peculiar phenomenon that the position in the (6) direction of the D5-branes does not play any role in the theory on the D3-brane, i.e., it is not a parameter of the three-dimensional theory.

我们注意到一个特殊现象:D5 膜在 (6) 方向的位置在 D3 膜上的理论中不起任何作用，也就是说，它不是三维理论的参数。

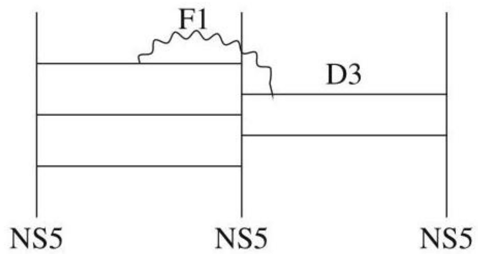
Note that we can also consider a D3-brane stretched between an NS5-brane and a D5-brane. The vector multiplet survives the projection of the former, whereas the hypermultiplet survives the projection of the latter. The theory thus contains only the supersymmetric vacuum.

注意，我们还可以研究伸展在一张 NS5 膜和一张 D5 膜之间的 D3 膜。矢量多重态会在前者的投影后保留，超多重态则在后者的投影后保留，因此该理论仅包含超对称真空。

Let us consider the following example:

我们来看下面这个例子:

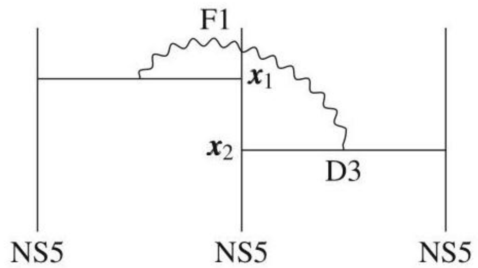
(62)




This describes the $U(3) \times U(2)$ gauge theory, with different gauge couplings for each factor. Let us now discuss the role of the fundamental string (F1) stretched from one D3-brane to the other across the NS5-brane. We consider the following system:

这描述了 $U(3) \times U(2)$ 规范理论，每个因子都有不同的规范耦合。现在我们来讨论跨 NS5 膜、从一张 D3 膜伸展到另一张 D3 膜的基本弦 (F1) 的作用。我们来研究下面这个系统:

(63)



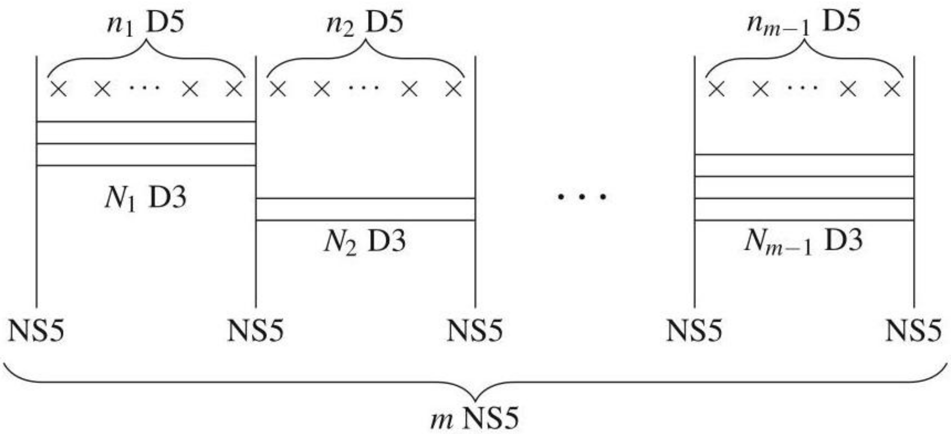

We expect that the F1 corresponds to a multiplet of mass $\mathbf{x}_1 - \mathbf{x}_2$ . We have to decide whether this is a vector multiplet or a hypermultiplet. Since the Chan-Paton factors of this fundamental string $\lambda_{iA}$ have an index $i$ of $\mathrm{U}(3)$ and an index $A$ of $\mathrm{U}(2)$ , the only reasonable possibility is the hypermultiplets transforming in the bifundamental representation $(\mathbf{3}, \mathbf{2})$ of $\mathrm{U}(3) \times \mathrm{U}(2)$ , whose mass is $\mathbf{x}_1 - \mathbf{x}_2$ . We will provide another argument for this shortly.

我们预期 F1 对应一个质量为 $\mathbf{x}_1 - \mathbf{x}_2$ 的多重态。我们需要判断这是矢量多重态还是超多重态。由于这根基本弦 $\lambda_{iA}$ 的陈-帕顿因子带有一个属于 $\mathrm{U}(3)$ 的指标 $i$ 和一个属于 $\mathrm{U}(2)$ 的指标 $A$ ，唯一合理的可能是它是双基本表示 $(\mathbf{3}, \mathbf{2})$ 下变换的超多重态，属于 $\mathrm{U}(3) \times \mathrm{U}(2)$ ，质量为 $\mathbf{x}_1 - \mathbf{x}_2$ 。我们很快会给出另一个论证。

More generally, we can consider the following system:

更一般地，我们可以研究下面这个系统:

(64)




The corresponding theory has $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{m-1})$ gauge group such that each $\mathrm{U}(N_k)$ factor has $n_k$ hypermultiplets (with $k = 1, \dots, m-1$ ) in the fundamental representation, and there are hypermultiplets in the representation $(\mathbf{N}_i, \mathbf{N}_{i+1})$ of the $\mathrm{U}(N_i) \times \mathrm{U}(N_{i+1})$ gauge group (with $i = 1, 2, \dots, m-2$ ). This gauge theory can be conveniently represented by the following diagram, known as the quiver diagram:

对应理论具有 $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_{m-1})$ 规范群，其中每个 $\mathrm{U}(N_k)$ 因子在基础表示下含有 $n_k$ 超多重子 (满足 $k = 1, \dots, m-1$ )，且该 $\mathrm{U}(N_i) \times \mathrm{U}(N_{i+1})$ 规范群的 $(\mathbf{N}_i, \mathbf{N}_{i+1})$ 表示下存在超多重子 (满足 $i = 1, 2, \dots, m-2$ )。该规范理论可以方便地用下图表示，也就是所谓的箭图:

(65)

$$\begin{tikzcd}[arrows=-] (N_1) \arrow[r] \arrow[d] & (N_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & (N_{m-1}) \arrow[d] \\ {[n_1]} & {[n_2]} & & {[n_{m-1}]} \end{tikzcd}$$

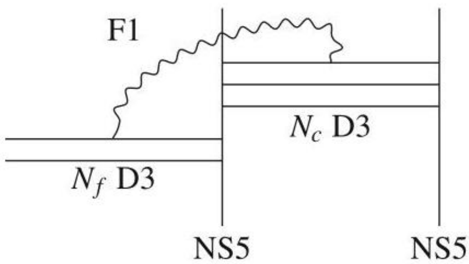
where each node $(k)$ with the round bracket denotes a $\mathrm{U}(k)$ gauge group, each node $[f]$ with the square brackets denotes $f$ flavors of hypermultiplets, and each line denotes a bifundamental hypermultiplet.

其中每个带圆括号的节点 $(k)$ 对应一个 $\mathrm{U}(k)$ 规范群，每个带方括号的节点 $[f]$ 对应 $f$ 味超多重子，每条线对应一个双基本超多重子。

Another way to introduce hypermultiplets in the fundamental representation is to put semi-infinite D3-branes, as depicted in the following diagram:

另一种引入基础表示超多重子的方法是放置半无限 D3 膜，如下图所示:

(66)




The fields on such branes live in a non-compact four-dimensional region, and so it is a non-dynamical background field from the perspective of a finite D3-brane. An open string stretched between a finite D3-brane and a semi-infinite D3-brane thus corresponds to a hypermultiplet in the fundamental representation of the gauge group on the finite D3-brane. The theory realized by the above diagram is the 3 d$\mathcal{N} = 4\mathrm{U}(N_c)$ gauge theory with $N_f$ hypermultiplets in the fundamental representation.

这类膜上的场存在于非紧致四维区域中，因此从有限 D3 膜的角度来看，它是一个非动力学背景场。因此，伸展在有限 D3 膜和半无限 D3 膜之间的开弦对应有限 D3 膜上规范群基础表示的超多重子。上图实现的理论是在基础表示下带有 $N_f$ 超多重子的 3 d$\mathcal{N} = 4\mathrm{U}(N_c)$ 规范理论。

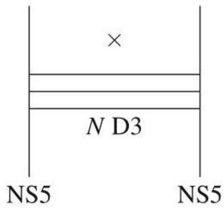
## Hanany-Witten Transition

## 花南-维滕跃迁

Let us consider the $U(N)$ gauge theory with one fundamental hypermultiplet. The corresponding brane configuration is

我们来考虑带有一个基本超多重子的 $U(N)$ 规范理论。对应的膜构型为

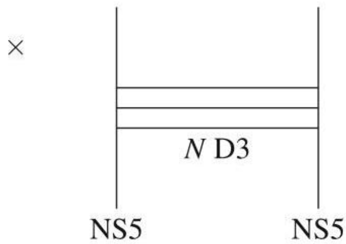
(67)




As we discuss before, the position of D5-brane in the (6) direction does not correspond to any parameter in the field theory. However, if we move the D5-brane as follows:

如我们之前讨论的，D5 膜在 (6) 方向的位置不对应场论中的任何参数。但如果我们按如下方式移动 D5 膜:

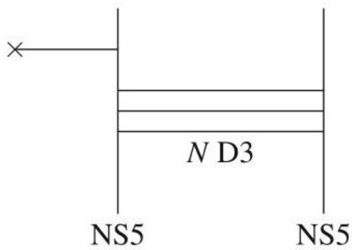
(68)




something happens. The 3-5 open strings are now certainly massive, and the mass depends on the position in the (6) direction, which is in contradiction with the above statement. In fact, when a D5-brane goes across an NS5-brane, there is the creation of a D3-brane that is stretched between the NS5-brane and the D5-brane. This phase transition is known as the Hanany-Witten transition [1]. In other words, the configuration (67) describes the same gauge theory as the following brane configuration:

发生了一些特殊变化。此时 3-5 开弦确实变为有质量的，质量依赖于其在 (6) 方向的位置，这和我们之前的结论矛盾。事实上，当一枚 D5 膜穿过 NS5 膜时，会生成一枚拉伸在 NS5 膜和 D5 膜之间的 D3 膜。这个相变就是我们熟知的花南-维滕跃迁 [1]。换句话说，构型 (67) 和如下膜构型描述的是同一个规范理论:

(69)

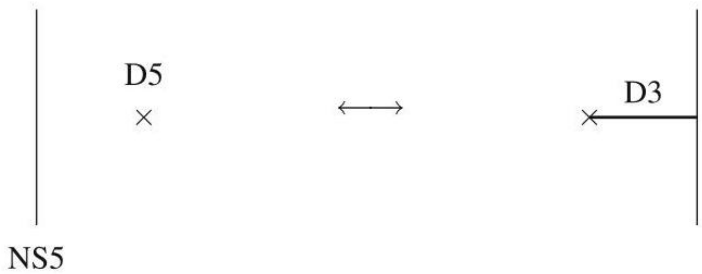


As discussed before, the D3-brane stretched between D5-brane and NS5-brane does not carry any field. The open string that is stretched between D3-branes within the interval and the external D3-brane corresponds to the hypermultiplet in the fundamental representation. If we send the D5-brane in (69) to minus infinity in the (6) direction, we recover the configuration (66) (with $N_c = N$ and $N_f = 1$ ). Thus, the configurations (67) and (69) as well as (66) (with $N_c = N$ and $N_f = 1$ ) describe the $\mathrm{U}(N)$ gauge theory with 1 flavor.

如之前讨论，拉伸在 D5 膜和 NS5 膜之间的 D3 膜不携带任何场。区间内的 D3 膜与外部 D3 之间拉伸的开弦对应基本表示的超多重子。如果我们将 (69) 中的 D5 膜沿 (6) 方向移到负无穷，我们就能得到构型 (66)(其中包含 $N_c = N$ 和 $N_f = 1$ )。因此，构型 (67)、(69) 以及 (带 $N_c = N$ 和 $N_f = 1$ 的) 构型 (66) 都描述 1 味的 $\mathrm{U}(N)$ 规范理论。

Let us discuss in further details the Hanany-Witten transition: every time a D5-brane crosses an NS5-brane, there is a D3-brane created and stretched between D5- and NS5-branes:

我们来更详细地讨论花南-维滕跃迁: 每当一枚 D5 膜穿过 NS5 膜, 就会生成一枚拉伸在 D5 膜和 NS5 膜之间的 D3 膜:




(70)

This is a consequence of the equations of motion [1]. Let us consider the field on the D5-brane. The $\mathrm{U}(1)$ field strength $F_{\mathrm{D5}}$ joins the two-form fields $B_{\mu\nu}^{\mathrm{NS}}$ associated with the NS5-brane via the linear combination $F_{\mathrm{D5}} - B^{\mathrm{NS}}$ , with an obvious change of notations compared to section "Branes Ending on Branes." As we discussed there, in order for the D3-brane to end on the D5-brane, we must have

这是运动方程的推论 [1]。我们来考虑 D5 膜上的场。$\mathrm{U}(1)$ 场强 $F_{\mathrm{D5}}$ 通过线性组合 $F_{\mathrm{D5}} - B^{\mathrm{NS}}$ 连接 NS5 膜对应的二形式场 $B_{\mu\nu}^{\mathrm{NS}}$ ，相较于“膜终止于膜”一节，这里做了记号调整。正如我们在那里讨论的，要让 D3 膜终止在 D5 膜上，我们必须满足

$$d\left(F_{\mathrm{D5}} - B^{\mathrm{NS}}\right) = \pm \sum \delta_{\mathrm{D3}}(\mathbf{y}), \tag{71}$$

where $y$ is a vector in the (789) directions and the sign depends on whether the D3-brane is on the right or on the left of the D5. (There is a $\theta\left(x^6\right)$ or $\theta\left(-x^6\right)$ multiplying the action for the D3-brane, which upon differentiation gives the difference in signs.) Note that $F_{\mathrm{D5}}$ is a nontrivial field (monopole) depending on the positions $\mathbf{y}$ in the (789) directions and the D3-branes are point-like objects in these directions. The equation can be rewritten as

其中 $y$ 是 (789) 方向的一个矢量，符号取决于 D3 膜位于 D5 膜的左侧还是右侧。(有一个 $\theta\left(x^{6}\right)$ 或 $\theta\left(-x^{6}\right)$ 乘在 D3 膜的作用量上，求导后就得到符号差。) 注意 $F_{\mathrm{D5}}$ 是依赖于 (789) 方向位置 $\mathbf{y}$ 的非平凡场 (单极子)，而 D3 膜在这些方向上是类点对象。该方程可以改写为

$$dF_{\mathrm{D5}} = H^{\mathrm{NS}} \pm \sum \delta_{\mathrm{D3}}\left(\mathbf{y}\right). \tag{72}$$

where $H^{\mathrm{NS}} = dB^{\mathrm{NS}}$ . Integrating this equation on $\mathbb{R}^{3}_{(789)}$ , we obtain

其中 $H^{\mathrm{NS}} = dB^{\mathrm{NS}}$ 。将这个方程在 $\mathbb{R}^{3}_{(789)}$ 上积分，我们得到

$$\int_{S^{2}} F_{\mathrm{D5}} = \int_{\mathbb{R}^{3}} H^{\mathrm{NS}} \pm \sum 1. \tag{73}$$

where the integral $\int_{\mathbb{R}^{3}} H^{\mathrm{NS}}$ is evaluated at the position of the D5-brane.

其中积分 $\int_{\mathbb{R}^{3}} H^{\mathrm{NS}}$ 是在 D5 膜的位置计算的。

An important fact is that the value of the total magnetic charge seen by the five-brane is measured at infinity and cannot be changed by moving the branes in space. In other words, the total charge at infinity is an invariant quantity. Therefore, the quantity

一个重要的结论是: 五膜观测到的总磁荷是在无穷远处测量的，无法通过在空间中移动膜改变。换句话说，无穷远处的总电荷是不变量。因此，这个量

$$\int_{\mathbb{R}^{3}_{(789)}} H^{\mathrm{NS}} + \text{ (\# D3 ending on the right) - (\# D3 ending on the left)} \tag{74}$$

must remain constant. We are interested in moving the D5-brane in the (6) direction. The quantity $\int_{\mathbb{R}^{3}} H^{\mathrm{NS}}$ depends on $x^{6}$ in the following way: Since an NS5-brane is a source of $H^{\mathrm{NS}}$ , we have $dH^{\mathrm{NS}} = \delta^{(6789)}$ . Taking the NS5-brane to be at $x^{6} = 0$ in the (6) direction, we have

必须保持恒定。我们感兴趣的是沿 (6) 方向移动 D5 膜。量 $\int_{\mathbb{R}^{3}} H^{\mathrm{NS}}$ 对 $x^{6}$ 的依赖关系如下: 由于 NS5 膜是 $H^{\mathrm{NS}}$ 的源，我们有 $dH^{\mathrm{NS}} = \delta^{(6789)}$ 。取 NS5 膜在 (6) 方向上位于 $x^{6} = 0$ 处，我们得到

$$\partial_{x^{6}} \int_{\mathbb{R}^{3}_{(789)}} H^{\mathrm{NS}} = \delta\left(x^{6}\right) \Rightarrow \int_{\mathbb{R}^{3}_{(789)}} H^{\mathrm{NS}} = \Theta\left(x^{6}\right), \tag{75}$$

where $\Theta\left(x^{6}\right) = 1/2$ for $x^{6} > 0$ and $\Theta\left(x^{6}\right) = -1/2$ for $x^{6} < 0$ . In particular, $\int_{\mathbb{R}^{3}_{(789)}} H^{\mathrm{NS}}$ jumps by one unit as one crosses the NS5-brane. Since (74) must remain constant, this must be compensated by the creation of a D3-brane. For example, in the left diagram of (70), we have $(74) = 1/2 + 0 - 0 = 1/2$ , whereas in the right diagram of (70) we have $(74) = -1/2 + 1 - 0 = 1/2$ .

其中 $\Theta\left(x^{6}\right) = 1/2$ 对应 $x^{6} > 0$ ，$\Theta\left(x^{6}\right) = -1/2$ 对应 $x^{6} < 0$ 。特别地，当穿过 NS5 膜时，$\int_{\mathbb{R}^{3}_{(789)}} H^{\mathrm{NS}}$ 会跳跃一个单位。由于 (74) 必须保持恒定，这个变化必须由 D3 膜的产生来补偿。例如，在 (70) 的左图中我们有 $(74) = 1/2 + 0 - 0 = 1/2$ ，而在 (70) 的右图中我们有 $(74) = -1/2 + 1 - 0 = 1/2$ 。

In general, the rule is that for every movement of the branes, the total magnetic charge of a D5-brane encoded in the linking number (with a conventional choice for the overall sign):

一般来说，规则是对于膜的任意移动，编码在环绕数中的 D5 膜总磁荷必须保持不变 (整体符号有常规选取):

$$\begin{aligned} L^{\mathrm{D5}} &= \frac{1}{2}\,(\text{\# NS5 on the right - \# NS5 on the left}) \\ &+ (\text{\# D3 on the left - \# D3 on the right}) \end{aligned} \tag{76}$$

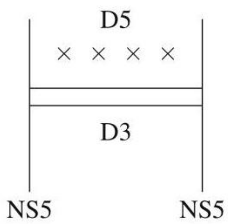
must remain invariant. By $S$ -duality, there is also a definition of the linking number for the NS5-brane; this is simply an exchange of D5 and NS5 in the above equation.

必须保持不变。通过 $S$ 对偶，NS5 膜也有环绕数的定义；只需要交换上述方程中的 D5 和 NS5 即可得到。

Let us now turn to the realizations of the Higgs and Coulomb branches from the brane configuration. As we discussed in section ”Application 1: Monopole Moduli Space,” the motion of the D3-branes in the (345) directions within the NS5-brane parametrizes the Coulomb branch of the theory. From the perspective of the NS5- brane, the D3-branes are monopoles, and so the Coulomb branch is identified with the moduli space of monopoles. We now consider the Higgs branch by looking at the example of the U(2) gauge theory with four hypermultiplets in the fundamental representation. A point of the Coulomb branch of such a theory can be realized by the following brane configuration:

现在我们来看从膜构型实现希格斯分支和库仑分支。正如我们在“应用 1: 单极模空间”一节中讨论的，D3 膜在 NS5 膜内部沿 (345) 方向的运动参数化了理论的库仑分支。从 NS5 膜的视角来看，D3 膜就是单极子，因此库仑分支等同于单极子的模空间。现在我们以带四个基本表示超多重子的 U(2) 规范理论为例讨论希格斯分支。这类理论库仑分支上的一点可以通过如下膜构型实现:

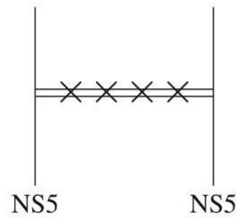
(77)




In order to go to the Higgs branch of this theory, all of the D3-branes must be coincident, and all of the masses must be zero. The corresponding configuration is

为了进入该理论的希格斯分支，所有 D3 膜必须重合，且所有质量必须为零。对应的构型是

(78)

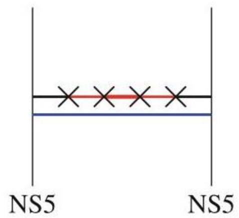



At this point, the D3-branes have the same position **x** along the (345) directions as the D5-branes and thus can move along the D5-branes. The Higgs branch corresponds to the D3-branes breaking in all possible ways into segments ending on pairs of D5-branes and moving in the directions orthogonal to the pictures. In field theory, the passage from the Coulomb branch to the Higgs branch corresponds to a change of boundary conditions. Indeed, when a D3-brane is stretched between two D5-branes, the former can move in the (789) directions, where its positions **y** correspond to the VEV of the scalar component of the hypermultiplet. Note that one can also realize a mixed Higgs-Coulomb branch, for example, from the following brane system:

此时，D3 膜与 D5 膜在 (345) 方向上具有相同位置 **x**，因此可以沿 D5 膜运动。希格斯分支对应 D3 膜以所有可能方式断裂为末端在 D5 膜对的片段，并在垂直于图的方向运动。在场论中，从库仑分支到希格斯分支的转变对应边界条件的改变。确实，当 D3 膜拉伸在两个 D5 膜之间时，D3 膜可以沿 (789) 方向运动，其位置 **y** 对应超多重子标量分量的真空期望值。注意我们也可以实现混合希格斯-库仑分支，例如来自如下膜系统:

(79)where the D3-brane segments in red can move along the D5-branes (corresponding to the motion in the Higgs branch), whereas the D3-brane segment in blue can move along the NS5-branes (corresponding to the motion in the Coulomb branch). The D3-brane segments in black with an end on the NS-branes cannot move.

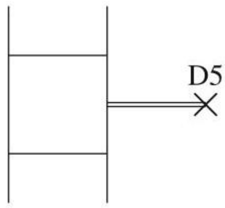
其中红色的 D3 膜片段可以沿 D5 膜运动 (对应希格斯分支上的运动)，蓝色的 D3 膜片段可以沿 NS5 膜运动 (对应库仑分支上的运动)。末端在 NS 膜上的黑色 D3 膜片段无法运动。




Upon stretching D3-branes between an NS5-brane and a D5-brane, there is an important rule, known as the $S$ -rule, that needs to be imposed for consistency. This rule states that there can be at most one D3-brane stretched between an NS5-brane and a D5-brane [1]. In fact, it was shown in [39] by applying a chain of T-dualities that the $S$ -rule is related to the Pauli exclusion principle in supersymmetric quantum mechanics. As an example, the following configuration violates the $S$ -rule:

当 D3 膜延伸在 NS5 膜和 D5 膜之间时，为了满足一致性，存在一条需要满足的重要规则，即 $S$ 规则。该规则指出，NS5 膜和 D5 膜之间最多只能延伸一根 D3 膜 [1]。事实上，文献 [39] 已经通过一系列 T 对偶变换证明，$S$ 规则与超对称量子力学中的泡利不相容原理相关。例如，以下构型就违反了 $S$ 规则:

(80)

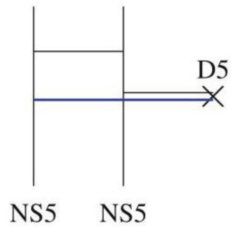



NS5 NS5
whereas the following configuration does not violate the $S$ -rule:

而以下构型不违反 $S$ 规则:

(81)where we highlighted in blue the D3-brane segment that is stretched from the leftmost NS5-brane to the D5-brane.

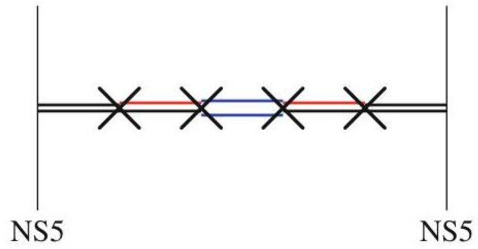
其中我们用蓝色标出了从最左侧 NS5 膜延伸到 D5 膜的 D3 膜段。




In our example of the U(2) gauge theory with four hypermultiplets, the quaternionic dimension of the Higgs branch is $(4 \times 2) - 4 = 4$ . (The quaternionic dimension is given by the number of hypermultiplets minus the number of vector multiplets: four constraints, or equivalently one in quaternionic units, need to be imposed on the hyperscalar fields for each generator of the gauge group, in order to take into account the three D-terms and the quotient by the corresponding element of the gauge group.) This means that there are four hypermultiplets at a generic point on the Higgs branch. We can see this number from the brane configuration by maximally splitting the D3-branes between D5-branes as follows and taking into account the $S$ -rule:

在我们讨论的带四个超多重子的 U(2) 规范理论例子中，希格斯分支的四元数维数是 $(4 \times 2) - 4 = 4$ 。(四元数维数等于超多重子数减去矢量多重子数: 为了考虑三个 D 项和规范群对应元素的商，需要对规范群每个生成元对应的超标量场施加四个约束，等价于四元数单位下的一个约束。) 这意味着在希格斯分支的一般点上存在四个超多重子。我们可以从膜构型中看出这个数: 如下所示将 D5 膜之间的 D3 膜最大拆分，并考虑 $S$ 规则:

(82)




The black D3-brane segments cannot move and do not correspond to any dynamical field. On the other hand, the blue and red D3-brane segments can move along the D5-branes in the (789) directions and hence correspond to the motion in the Higgs branch. Observe that there are in total four of them (two blue segments in the middle + one red segment on the left + one red segment on the right), in agreement with the quaternionic dimension of the Higgs branch.

黑色的 D3 膜段无法移动，不对应任何动力学场。另一方面，蓝色和红色的 D3 膜段可以沿 D5 膜在 (789) 方向运动，因此对应希格斯分支上的运动。可以看到总共有四个可动段 (中间两段蓝色 + 左侧一段红色 + 右侧一段红色)，与希格斯分支的四元数维数一致。

## Application 2: Three-Dimensional Mirror Symmetry

## 应用 2: 三维镜像对称

Let us first discuss this from the field theory perspective [2]. As we mentioned before, the hypermultiplets and vector multiplets are indistinguishable. One could ask the following question: given a $3d\mathcal{N} = 4$ gauge theory, is there a ”mirror theory” whose Coulomb branch is equal to the Higgs branch of the original theory and vice-versa? This duality can exist only in three dimensions.

我们先从场论角度讨论这个问题 [2]。如前文所述，超多重子与矢量多重子是不可区分的。我们可以提出这样一个问题: 给定一个 $3d\mathcal{N} = 4$ 规范理论，是否存在一个“镜像理论”，其库仑分支等于原理论的希格斯分支，反之亦然？这种对偶性仅存在于三维中。

Since $\mathrm{SU}(2)_V$ acts on the scalars parametrizing the Coulomb branch and $\mathrm{SU}(2)_H$ acts on the scalars of the hypermultiplets, $\mathrm{SU}(2)_V$ must be interchanged with $\mathrm{SU}(2)_H$ .

由于 $\mathrm{SU}(2)_V$ 作用在参数化库仑分支的标量上，$\mathrm{SU}(2)_H$ 作用在超多重子的标量上，因此 $\mathrm{SU}(2)_V$ 必须与 $\mathrm{SU}(2)_H$ 互换。

The parameters in this theory are the masses $\mathbf{m}$ of the hypermultiplets, transforming under $\mathrm{SU}(2)_V$ , and the FI parameters $\zeta$ , transforming under $\mathrm{SU}(2)_H$ . We see from (35) in section ” $\mathcal{N} = 2$ Gauge Theories in Four Dimensions” that the FI parameters $\zeta$ imply a VEV for the hyperscalars and generally break the gauge symmetry. Moreover, as background fields on the Higgs branch, the components of $\zeta$ lift part or the whole Coulomb branch of the theory.

该理论中的参数是超多重子的质量 $\mathbf{m}$ ，在 $\mathrm{SU}(2)_V$ 下变换；以及 FI 参数 $\zeta$ ，在 $\mathrm{SU}(2)_H$ 下变换。我们从“$\mathcal{N}=2$ 四维规范理论”一节的 (35) 式中可以看到，FI 参数 $\zeta$ 给出了超标的真空期望值，通常会破缺规范对称性。此外，作为希格斯分支上的背景场，$\zeta$ 的分量会抬高该理论的部分或整个库仑分支。

Summarizing, the duality exchanges the following objects in the original and mirror theories: (1) the Higgs branch and Coulomb branch,(2) the $\mathrm{SU}(2)_H$ and $\mathrm{SU}(2)_V R$ -symmetries, and (3) the mass parameters $\mathbf{m}$ and FI parameters $\boldsymbol{\zeta}$ .

综上，对偶性在原理论和镜像理论之间互换了以下对象:(1) 希格斯分支与库仑分支，(2) $\mathrm{SU}(2)_H$ 对称性与 $\mathrm{SU}(2)_V R$ 对称性，(3) 质量参数 $\mathbf{m}$ 与 FI 参数 $\boldsymbol{\zeta}$ 。

Let us now discuss an example of mirror theories [2]. We consider the U(1) gauge theory with $N$ hypermultiplets of charge $1$ . For convenience, we refer to this as Theory A. The mirror theory is described by a circular quiver with $\prod_{i=1}^{N} \mathrm{U}(1)_i$ gauge groups with hypermultiplets of charge $(+1,-1)$ under $\mathrm{U}(1)_i \times \mathrm{U}(1)_{i+1}$ , where due to cyclicity we identify $\mathrm{U}(1)_{N+1} \equiv \mathrm{U}(1)_1$ . We refer to this as theory B. Note that there is an overall $\mathrm{U}(1)$ that acts trivially on the matter fields and this needs to be modded out. Upon removing an overall $\mathrm{U}(1)$ , an equivalent way to represent theory B is a linear quiver with $\prod_{j=1}^{N-1} \mathrm{U}(1)_j$ gauge group with the following matter content:

现在我们来讨论一个镜像理论的例子 [2]。我们考虑带有 $N$ 个电荷为 1 的超多重子的 U(1) 规范理论，为方便起见，我们称之为理论 A。其镜像理论由一个圆形箭袋描述，带有 $\prod_{i=1}^{N} \mathrm{U}(1)_i$ 个规范群，超多重子在 $\mathrm{U}(1)_i \times \mathrm{U}(1)_{i+1}$ 下带电荷 $(+1,-1)$ ，由于循环性我们约定 $\mathrm{U}(1)_{N+1} \equiv \mathrm{U}(1)_1$ ，我们称之为理论 B。注意存在一个整体的 $\mathrm{U}(1)$ ，它对物质场的作用是平凡的，需要将其商去。去掉整体的 $\mathrm{U}(1)$ 后，理论 B 的等价表述是一个带有 $\prod_{j=1}^{N-1} \mathrm{U}(1)_j$ 个规范群的线性箭袋，其物质内容如下:

- The hypermultiplets of charge $(+1,-1)$ under $\mathrm{U}(1)_j \times \mathrm{U}(1)_{j+1}$

- 在 $\mathrm{U}(1)_j \times \mathrm{U}(1)_{j+1}$ 下带电荷 $(+1,-1)$ 的超多重子

- One hypermultiplet of charge 1 under $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_{N-1}$ gauge groups

- 一个同时在 $\mathrm{U}(1)_1$ 和 $\mathrm{U}(1)_{N-1}$ 规范群下带电荷 1 的超多重子

These theories can be depicted diagrammatically as

这些理论可以用图表示为

(83)

$$(1) - [N] \qquad\qquad [1] - \overbrace{(1) - (1) - \cdots - (1)}^{N-1 \text{ gauge groups}} - [1]$$

Theory A $\qquad\qquad$ Theory B

The quaternionic dimensions of the Higgs and Coulomb branches and the number of parameters are

希格斯分支与库仑分支的四元数维数以及参数个数为

(84)

| Theory | $\dim_{\mathbb{H}}$ Higgs | $\dim_{\mathbb{H}}$ Coulomb | #FI | #masses |
|---|---|---|---|---|
| A | $N-1$ | 1 | 1 | $N-1$ |
| B | 1 | $N-1$ | $N-1$ | 1 |

where the quaternionic Higgs branch dimensions can be obtained from the total numbers of hypermultiplets minus the total dimension of the gauge symmetry and the quaternionic Coulomb branch dimension is the total rank of the gauge symmetry. Note that for every U(1) factor, one can eliminate a mass, since the effective parameter is $\mathbf{m} - \mathbf{x}$ , and so by redefining the origin of $\mathbf{x}$ (which can be done only for $\mathrm{U}(1)$ ) we can set $\mathbf{m}$ to zero. For this reason, theory A has $N-1$ independent mass parameters, and theory B has one independent mass parameter.

其中四元数希格斯分支的维数可由超多重子总数减去规范对称的总维数得到，而四元数库仑分支的维数等于规范对称的总秩。注意，对于每一个 U(1) 因子，我们都可以消去一个质量，因为有效参数是 $\mathbf{m}-\mathbf{x}$ ，因此通过重新定义 $\mathbf{x}$ 的原点 (这仅对 $\mathrm{U}(1)$ 可行)，我们可以将 $\mathbf{m}$ 设为零。因此，理论 A 有 $N-1$ 个独立质量参数，理论 B 有一个独立质量参数。

One can check that the moduli space of the two theories is the same and the Higgs and Coulomb branches are exchanged. For example, we can easily see that the Higgs branch of theory B coincides with the Coulomb branch of theory A.

可以验证，两个理论的模空间是相同的，且希格斯分支与库仑分支互换。例如，我们可以很容易看出，理论 B 的希格斯分支与理论 A 的库仑分支重合。

The circular quiver description of theory B is in fact the Dynkin diagram of the affine $A_{N-1}$ Lie algebra. As we already discussed in section " $\mathcal{N}=2$ Gauge Theories in Four Dimensions," this quiver appears in the work of Kronheimer [21]. The Higgs branch of the theory is given by the $\mathbf{D}$ -term equations divided by the $\mathrm{U}(1)^{N-1}$ gauge symmetry. This gives rise to the algebraic equations that define the ALE space $\mathbb{C}^2/\mathbb{Z}_N$ . The $N-1$ FI parameters of the theory correspond to the $N-1$ parameters of the blowup of the ALE space.

理论 B 的环形箭图描述实际上就是仿射 $A_{N-1}$ 李代数的邓肯图。正如我们在章节“$\mathcal{N}=2$ 四维规范理论”中已经讨论论过的，这个箭图出现在 Kronheimer 的工作 [21] 中。该理论的希格斯分支由 $\mathbf{D}$ 项方程除以 $\mathrm{U}(1)^{N-1}$ 规范对称性得到，由此生成定义 ALE 空间 $\mathbb{C}^2/\mathbb{Z}_N$ 的代数方程。该理论的 $N-1$ 个 FI 参数对应 ALE 空间爆破的 $N-1$ 个参数。

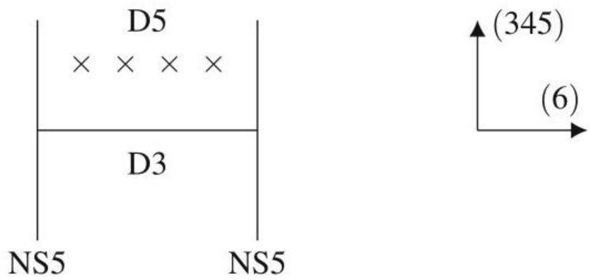
We emphasize that this duality holds only in the infrared. Since we are working in three dimensions and the coupling $g^2$ has the dimensions of mass, it follows that the infrared corresponds to the strong coupling regime. (The dimensionless constant is in fact $\tilde{g}^2 = g^2/\mu^2$ where $\mu$ is the energy scale at which we observe the theory. Going to the infrared means taking $\mu \to 0$ which corresponds to taking $\tilde{g} \to \infty$ , for a fixed dimensionful quantity $g$ .) We have computed the metric of the Coulomb branch of theory A in section "$\mathcal{N} = 4$ Gauge Theories in Three Dimensions," which is identified as the Taub-NUT space. For $1/g_{\text{cl}}^2 = 0$ , this becomes the metric of the ALE space associated with $\mathbb{C}^2/\mathbb{Z}_N$ , which is the Higgs branch of theory $B$ in agreement with the result of Kronheimer. (Indeed, a possible way of understanding this duality is to observe that theory A can be realized on a system of one D2 and $N$ D6-branes in type IIA and theory B on the worldvolume of a D2-brane probing a $\mathbb{C}^2/\mathbb{Z}_N$ singularity again in type IIA. Both theories uplift to the same M theory realization, an M2-brane at a $\mathbb{C}^2/\mathbb{Z}_N$ singularity, explaining the IR duality [40].)

我们强调，这种对偶性仅在红外区成立。由于我们在三维中研究，耦合 $g^2$ 具有质量量纲，因此红外对应强耦合区。(无量纲常数实际上是 $\tilde{g}^2 = g^2/\mu^2$ ，其中 $\mu$ 是我们观测理论的能标。趋近红外意味着取 $\mu \to 0$ ，对应固定有量纲量 $g$ 时取 $\tilde{g} \to \infty$ 。) 我们已经在章节“$\mathcal{N} = 4$ 三维规范理论”中计算了理论 A 库仑分支的度量，它被识别为陶布-纽特空间。对 $1/g_{\text{cl}}^2 = 0$ ，该度量成为与 $\mathbb{C}^2/\mathbb{Z}_N$ 关联的 ALE 空间的度量，也就是理论 $B$ 的希格斯分支，与 Kronheimer 的结果一致。(事实上，理解这种对偶性的一种可能思路是: 理论 A 可在 IIA 型弦论中由一个 D2 膜和 $N$ 个 D6 膜系统实现，理论 B 则同样在 IIA 型弦论中由探测 $\mathbb{C}^2/\mathbb{Z}_N$ 奇点的 D2 膜世界体积实现。两种理论都 uplift 为同一个 M 理论实现: 位于 $\mathbb{C}^2/\mathbb{Z}_N$ 奇点处的 M2 膜，这就解释了红外对偶性 [40]。)

From the brane perspective, theory A can be realized as follows (we take $N = 4$ for simplicity):

从膜的视角看，理论 A 可以按如下方式构造 (为简化我们取 $N = 4$ ):

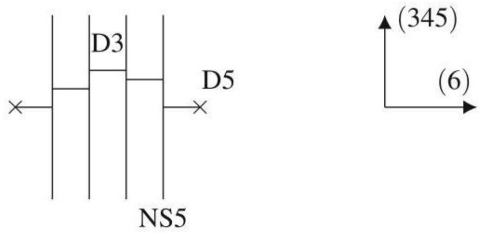
(85)




Note that the NS5-brane is a point in the directions (6; 789). Let us now apply $S$ -duality, which exchanges D5- and NS5-branes to the above brane system. Now, the D5-brane is a point in the directions (6; 789). In order to reconcile with our original setup, where the D5-brane were points in the directions (6; 345), we need to accompany $S$ -duality by an interchange of the (345) directions and the (789) directions. The resulting theory is then

注意，NS5 膜在方向 (6;789) 中是一个点。现在对上述膜系统应用 $S$ 对偶，该变换互换 D5 膜与 NS5 膜。变换后，D5 膜在方向 (6;789) 中是一个点。为了匹配我们最初的设置 (即 D5 膜是方向 (6;345) 中的点)，我们需要在 $S$ 对偶后交换 (345) 方向与 (789) 方向，最终得到的理论为

(86)

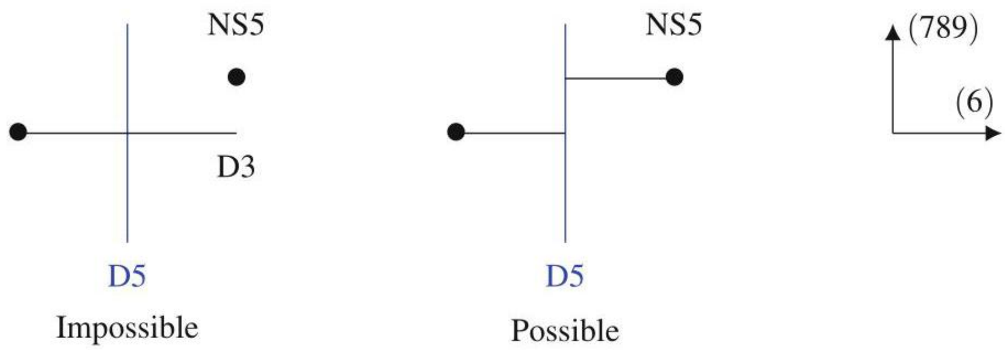



This indeed realizes theory B, with the description [1] - (1) - (1) - (1) - [1], as required. Mirror symmetry can therefore be realized as $S$ -duality along with the exchange (345) ↔ (789). The latter corresponds to the exchange of $SU(2)_V$ and $SU(2)_H R$ -symmetries.

这确实按照要求实现了理论 B，描述为 [1] - (1) - (1) - (1) - [1]。因此镜像对称性可以实现为 $S$ 对偶，同时伴随交换 (345) ↔ (789)。后者对应 $SU(2)_V$ 与 $SU(2)_H R$ 对称性的交换。

Let us now see what happens to the mass and FI parameters. As we discussed, the mass parameters correspond to the relative positions of the D5-branes in the (345) directions. Similarly, the FI parameters correspond to the relative positions $\Delta \mathbf{y}_{\mathrm{NS}} \equiv \mathbf{y}_{\mathrm{NS}_1} - \mathbf{y}_{\mathrm{NS}_2} \neq 0$ of two NS5-branes in the (789) directions. Indeed, we see that if $\Delta \mathbf{y}_{\mathrm{NS}} \neq 0$ , a D3-brane cannot be stretched between such two NS5-branes, and so the theory does not have a supersymmetric vacuum. Indeed, as expected from the field theory, the FI parameter lifts the supersymmetric vacuum of the U(1) pure gauge theory. It is thus natural to identify $\Delta \mathbf{y}_{\mathrm{NS}}$ as the FI parameter. As a further test of this statement, we can add a hypermultiplet which corresponds to adding a D5-brane into the system. Now, equation (35) can be solved and we expect supersymmetric vacua. Let us draw the corresponding brane diagram in such a way that the vertical direction is (789), where the NS5-branes are points and the D5- brane is denoted by the blue vertical line:

现在我们来看质量参数和 FI 参数会如何变化。正如我们讨论过的，质量参数对应 D5 膜在 (345) 方向的相对位置。类似地，FI 参数对应两个 NS5 膜在 (789) 方向的相对位置 $\Delta \mathbf{y}_{\mathrm{NS}} \equiv \mathbf{y}_{\mathrm{NS}_1} - \mathbf{y}_{\mathrm{NS}_2} \neq 0$ 。不难发现，如果 $\Delta \mathbf{y}_{\mathrm{NS}} \neq 0$ ，D3 膜无法在这两个 NS5 膜之间拉伸，因此理论不存在超对称真空。这正如场论所预期的那样，FI 参数抬高了 U(1) 纯规范理论的超对称真空。因此将 $\Delta \mathbf{y}_{\mathrm{NS}}$ 认作 FI 参数是自然的。为进一步验证这个结论，我们可以加入一个超多重子，对应往系统中加入一个 D5 膜。此时方程 (35) 有解，我们预期存在超对称真空。我们来绘制对应的膜图: 设定竖直方向为 (789) 方向，NS5 膜在图中为点，D5 膜用蓝色竖线表示:




(87)

As we can see from the above diagrams, in order for the D3-brane to be able to stretch between two NS5-branes, one needs to split the former in two segments along the NS5-branes. The relative positions of the two segments in the (789) directions are indeed equal to $\Delta\mathbf{y}_{\mathrm{NS}}$ . Therefore, we should indeed interpret the relative positions between two NS5-branes in the (789) directions as the FI parameters, which force the hypermultiplet scalars to have a VEV. The effect of the $S$ -duality is as expected, namely, it exchanges the mass parameters (corresponding to the D5-positions) and the FI parameters (corresponding to the NS5-positions).

从上图可以看出，为了让 D3 膜能够在两个 NS5 膜之间拉伸，需要将 D3 膜沿着 NS5 膜拆分为两段。这两段在 (789) 方向的相对位置确实等于 $\Delta\mathbf{y}_{\mathrm{NS}}$ 。因此，我们确实应当将两个 NS5 膜在 (789) 方向的相对位置诠释为 FI 参数，这类参数要求超多重子标量获得真空期望值。$S$ 对偶的效应也符合预期: 它交换了质量参数 (对应 D5 膜的位置) 和 FI 参数 (对应 NS5 膜的位置)。

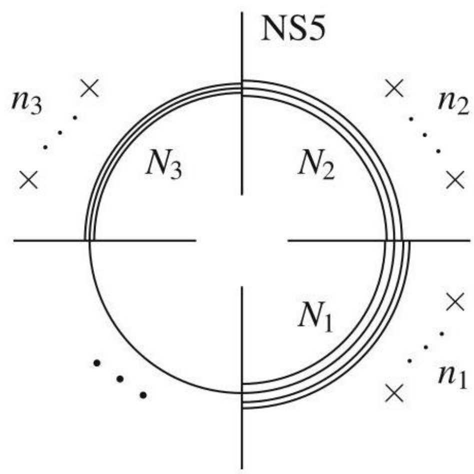
## Application 3: Circular Quivers and the ABJM Theory

## 应用 3: 环形箭形与 ABJM 理论

We can also consider a setup where the direction (6) is compactified on a circle. We have an array of NS5-branes sitting at distinct points on the circle and D3- branes stretched among them. The configuration is as in Figure (64) where now the rightmost NS5-brane is identified with the leftmost one. We depict this configuration below:

我们也可以考虑将 6 维方向紧致化到一个圆上的构造: 一系列 NS5 膜位于圆上的不同点，D3 膜在这些膜之间延伸。该构造如图 (64) 所示，其中最右侧的 NS5 膜与最左侧的 NS5 膜等价，我们在下方绘制该构造:

(88)



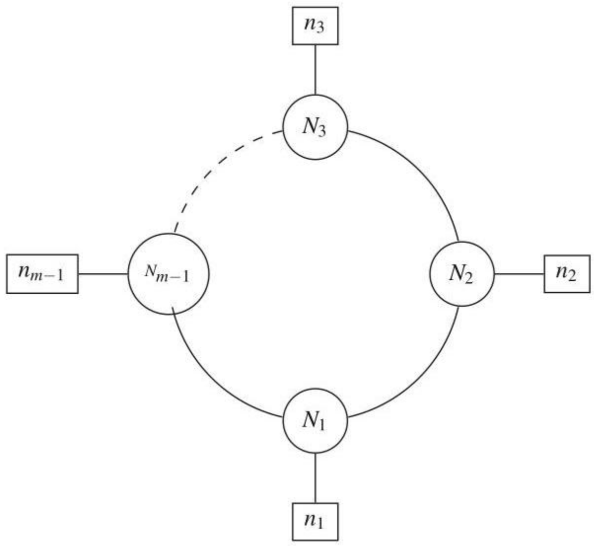

where the arcs with label $N$ denote $N$ D3-branes, each straight line in the radial direction denotes an NS5-brane, and the crosses with label $n$ denote $n$ D5-branes.

其中标记为 $N$ 的弧线表示 $N$ 张 D3 膜，径向的每条直线表示一张 NS5 膜，标记为 $n$ 的十字表示 $n$ 张 D5 膜。

The corresponding theory has $\mathrm{U}(N_1)\times\mathrm{U}(N_2)\times\cdots\times\mathrm{U}(N_{m-1})$ gauge group such that each $\mathrm{U}(N_k)$ factor has $n_k$ hypermultiplets (with $k=1,\ldots,m-1$ ) in the fundamental representation, and there are hypermultiplets in the representation $(\mathbf{N}_i,\mathbf{N}_{i+1})$ of the $\mathrm{U}(N_i)\times\mathrm{U}(N_{i+1})$ gauge group (with $i=1,2,\ldots,m-1$ ), where due to cyclicity we identify $\mathrm{U}(N_m)\equiv\mathrm{U}(N_1)$ . This can be depicted as follows:

对应的理论具有 $\mathrm{U}(N_1)\times\mathrm{U}(N_2)\times\cdots\times\mathrm{U}(N_{m-1})$ 个规范群，每个 $\mathrm{U}(N_k)$ 因子在基础表示下有 $n_k$ 个超多重子 (满足 $k=1,\ldots,m-1$ )，同时在 $\mathrm{U}(N_i)\times\mathrm{U}(N_{i+1})$ 规范群的表示 $(\mathbf{N}_i,\mathbf{N}_{i+1})$ 下存在超多重子 (满足 $i=1,2,\ldots,m-1$ )，由于循环性我们规定 $\mathrm{U}(N_m)\equiv\mathrm{U}(N_1)$ 等价。这可以绘制为:

(89)




This is the Kronheimer-Nakajima quiver [41] whose Higgs branch describes the moduli space of instantons on ALE spaces of type $A$ .

这是克罗海默-中岛箭形 [41]，其希格斯分支描述了 $A$ 型 ALE 空间上的瞬子模空间。

We already encountered the quiver with $n_k=0$ and $N_k=N$ , with $k=1,\ldots\ m-1$ , as the worldvolume theory of $N$ D2-branes at the orbifold $\mathbb{R}^4/\mathbb{Z}_{m-1}$ . This is not a coincidence. A T-duality along a compactified direction along the singularity converts D2-branes into D3-branes wrapping the circle and the singularity itself into $m-1$ NS5-branes [42].

我们已经遇到过 $n_k=0$ 和 $N_k=N$ 满足 $k=1,\ldots\ m-1$ 的箭形，它是轨道 $\mathbb{R}^4/\mathbb{Z}_{m-1}$ 处 $N$ 张 D2 膜的世界体积理论，这并非巧合。沿着奇点方向的紧致化方向做 T 对偶，会将 D2 膜转变为缠绕圆的 D3 膜，将奇点本身转变为 $m-1$ 张 NS5 膜 [42]。

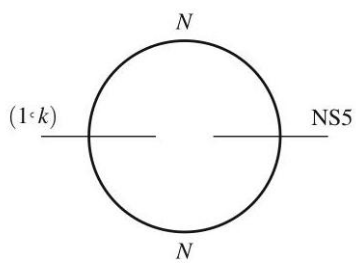
Making a slight detour, we notice that we can generalize these configurations by introducing $(1,k)$ 5-branes. (Recall that type IIB is invariant under $\mathrm{SL}(2,\mathbb{Z})$ transformations associated with S-duality. There exist $(p,q)$ 5-branes, which can be considered as a bound state of $p$ NS5 and $q$ D5-branes, transforming as a multiplet of $\mathrm{SL}(2,\mathbb{Z})$ . The $S$ generator inverts the gauge coupling and exchanges D5- and NS5-branes, while the $T$ generator shifts the axion $C_0$ and converts a NS5-branes into an $(1,1)$ 5-brane.) Supersymmetry requires that the $(1,k)$ 5-branes are rotated in the (345789) directions [43,44]. By rotating the $(1,k)$ 5 -branes by the common angle $\theta = \arctan k$ in the planes $(3,7)$, $(4,8)$ , and $(5,9)$ , we can preserve $\mathcal{N} = 3$ supersymmetry. This introduces an $\mathcal{N} = 3$ preserving Chern-Simons terms for the D3-branes ending on the $(1,k)$ 5-brane.

稍微拓展一下，我们可以通过引入 $(1,k)$ 5 膜来推广这些构造。(回忆一下，IIB 型理论在与 S 对偶相关的 $\mathrm{SL}(2,\mathbb{Z})$ 变换下不变。存在 $(p,q)$ 5 膜，可以看作是 $p$ 张 NS5 膜和 $q$ 张 D5 膜的束缚态，在 $\mathrm{SL}(2,\mathbb{Z})$ 的多重态下变换。$S$ 生成元反转规范耦合，交换 D5 膜与 NS5 膜；$T$ 生成元移动轴子 $C_0$ ，将 NS5 膜转变为 $(1,1)$ 5 膜。) 超对称要求 $(1,k)$ 5 膜在 (345789) 方向旋转 [43,44]。通过将 $(1,k)$ 5 膜在平面 $(3,7)$, $(4,8)$ 和 $(5,9)$ 中旋转公共角度 $\theta = \arctan k$ ，我们可以保留 $\mathcal{N} = 3$ 超对称，这会为终止于 $(1,k)$ 5 膜的 D3 膜引入一个保留 $\mathcal{N} = 3$ 超对称的陈-西蒙斯项。

A particularly interesting configuration is obtained by considering a circular quiver with one NS5 and one $(1,k)$ 5-brane, depicted below:

通过考虑一个包含一张 NS5 膜和一张 $(1,k)$ 5 膜的环形箭袋，我们得到了一个特别有趣的构型，如下图所示:

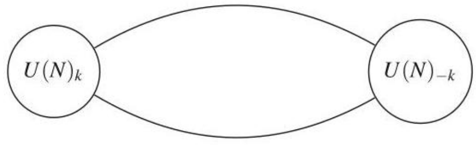
(90)




The resulting $\mathcal{N} = 3$ Chern-Simons-Yang-Mills theory has gauge group $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ , and it is coupled to two bifundamental hypermultiplets, where the subscripts $k$ and $-k$ denote the Chern-Simons levels.

由此得到的 $\mathcal{N} = 3$ 陈-西蒙斯-杨-米尔斯理论的规范群为 $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ，它耦合了两个双基本超多重子，其中下标 $k$ 和 $-k$ 表示陈-西蒙斯能级。

(91)




It is argued in [45] that the theory flows in the IR to a $\mathcal{N} = 6$ superconformal theory, a Chern-Simons-matter theory obtained by neglecting the Yang-Mills terms. This is called the ABJM theory. It is also argued in [45] that the same theory arises as the IR limit of a set of $N$ M2-branes probing a $\mathbb{R}^8/\mathbb{Z}_k$ singularity in M-theory.

文献 [45] 指出，该理论在红外区流向 $\mathcal{N}=6$ 超共形理论——这是忽略杨-米尔斯项后得到的陈-西蒙斯-物质理论，也就是 ABJM 理论。文献 [45] 还指出，该理论也可以由 M 理论中探测 $\mathbb{R}^8/\mathbb{Z}_k$ 奇点的 $N$ 张 M2 膜的红外极限得到。

In particular, for $k=1$ , the theory flows to a superconformal theory with maximal $\mathcal{N}=8$ supersymmetry. We can see the ABJM theory with $k=1$ as the IR limit of both the $\mathcal{N}=8$ SYM theory in three dimensions or, alternatively, as the IR limit of the worldvolume theory of $N$ M2-branes in flat space. These statements follow from maximal supersymmetry, but they can be also seen by the following brane argument [45]. For $k=1$ , we have a circular quiver with an NS5- and a (1, 1)5-brane. Using S-duality, which leaves the D3-branes unaltered, we can convert it to a circular quiver with one D5- and one $(1,1)\,5$ -brane. With a shift in the axion (a $T$ transformation which preserves the D5), we finally obtain a circular quiver with one D5- and one NS5-brane. This corresponds to a $\mathcal{N}=4$ theory with gauge group $\mathrm{U}(N)$ coupled to one adjoint and one fundamental hypermultiplet. The same theory can be also realized on a set of $N$ D2-branes and a single D6-brane, as discussed in section " $\mathcal{N}=4$ Gauge Theories in Three Dimensions". We can study the IR limit of this theory by uplifting to M-theory, where the D6-branes become an ALE space $\mathbb{R}^4/\mathbb{Z}_1$ which is just a flat space. We see that the fundamental hypermultiplet has no effect in the IR, and we are left with the same physics of a set of $N$ D2-branes in type IIA, which supports an $\mathcal{N}=8$ SYM theory with group $\mathrm{U}(N)$ on the worldvolume and whose uplift to M-theory gives the $\mathcal{N}=8$ theory living on M2-branes.

特别地，对于 $k=1$ ，该理论流向具有最大 $\mathcal{N}=8$ 超对称的超共形理论。我们可以看到，具有 $k=1$ 的 ABJM 理论既是三维 $\mathcal{N}=8$ SYM 理论的红外极限，也可以是平坦空间中 $N$ 张 M2 膜世界体积理论的红外极限。这些结论由最大超对称性导出，也可以通过下面的膜论证得到 [45]。对于 $k=1$ ，我们得到包含一张 NS5 膜和一张 (1,1)5 膜的环形箭袋。利用不改变 D3 膜的 S 对偶，我们可以将其转化为包含一张 D5 膜和一张 $(1,1)\,5$ 膜的环形箭袋。通过轴子移位 (一种保留 D5 膜的 $T$ 变换)，我们最终得到包含一张 D5 膜和一张 NS5 膜的环形箭袋。这对应一个规范群为 $\mathrm{U}(N)$ 的 $\mathcal{N}=4$ 理论，耦合了一个伴随超多重子和一个基本超多重子。正如“三维 $\mathcal{N}=4$ 规范理论”一节所述，该理论也可以由一组 $N$ 张 D2 膜和一张单独的 D6 膜实现。我们可以通过 uplift 到 M 理论来研究该理论的红外极限，在 M 理论中 D6 膜成为 ALE 空间 $\mathbb{R}^4/\mathbb{Z}_1$ ，也就是平坦空间。我们可以发现基本超多重子在红外区没有影响，最终得到的物理和 IIA 理论中一组 $N$ 张 D2 膜的物理一致，对应世界体积上具有规范群 $\mathrm{U}(N)$ 的 $\mathcal{N}=8$ SYM 理论，将其 uplift 到 M 理论就得到 M2 膜上的 $\mathcal{N}=8$ 理论。

## Hanany-Witten Brane Configurations in Four Dimensions

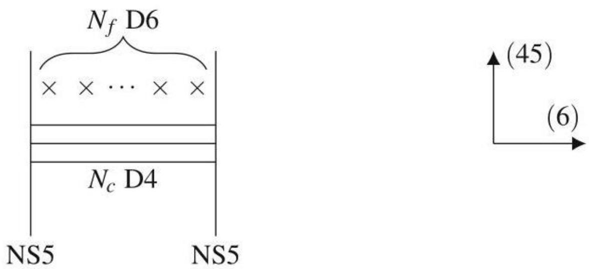
## 四维中的 Hanany-Witten 膜构型

Starting from the Hanany-Witten configuration, we can obtain the brane configuration that realizes a $4d\mathcal{N}=2$ gauge theory by applying a $T$ -duality along the direction (3). We are now in type IIA string theory: the D3-branes turn into D4- branes, and the D5-branes turn into D6-branes, whereas the NS5-branes remain NS5-branes. The reason for the latter is because the NS5-brane is an object associated with closed strings, and so it is invariant under $T$ -duality along its worldvolume. Under $T$ -duality, the NS5-brane in type IIB becomes the NS5-brane in type IIA, which has a tensor multiplet on the worldvolume and not a vector multiplet.

从 Hanany-Witten 构型出发，我们可以通过沿方向 (3) 应用 $4d\mathcal{N}=2$ 对偶得到实现 $T$ 规范理论的膜构型。我们现在处于 IIA 型弦论中:D3 膜变为 D4 膜，D5 膜变为 D6 膜，而 NS5 膜仍为 NS5 膜。后者的原因是 NS5 膜是与闭弦相关的客体，因此它在沿其世界体积的 $T$ 对偶下不变。在 $T$ 对偶下，IIB 中的 NS5 膜变为 IIA 中的 NS5 膜，其世界体积上是张量多重态而非矢量多重态。

The configuration is very similar to that of Hanany-Witten [46]. For example, the 4 $\mathrm{d}\mathcal{N}=2\mathrm{U}(N)$ gauge theory with $N_f$ flavors of hypermultiplets in the fundamental representation is realized by the following system:

该构型与文献 [46] 中的 Hanany-Witten 构型非常相似。例如，带有 $N_f$ 个基础表示超多重态味的 4 $\mathrm{d}\mathcal{N}=2\mathrm{U}(N)$ 规范理论可由如下系统实现:




(92)

The Coulomb branch of this theory is parametrized by a complex scalar field corresponding to the position $v$ of the D4-branes in the (45) directions along the NS5-branes. The vector multiplet contains the gauge field $A_\mu$ and $v$ , where $A_\mu$ cannot be dualized. The D6-brane, which spans the (0123789) directions, is now a point in the (45) directions. The relative positions of the D6-branes in (45) correspond to the complex masses of the hypermultiplets.

该理论的库仑分支由一个复标量场参数化，对应于 D4 膜在 NS5 膜之间 (45) 方向的位置 $v$ 。矢量多重态包含规范场 $A_\mu$ 和 $v$ ，其中 $A_\mu$ 无法被对偶化。覆盖 (0123789) 方向的 D6 膜在 (45) 方向是一个点，D6 膜在 (45) 方向的相对位置对应超多重态的复质量。

The $S$ -rule and the fact that if the D6-brane crosses the NS5-brane there is a D4- brane creation are still valid. However, there is a substantial difference with type IIB construction. In type IIB string theory, we know that the term responsible for the ending of the D3-brane on the NS5 is $\int F\wedge A^{(4)}$ . (One can calculate it from the ending on a D5-brane and perform S-duality under which $A^{(4)}$ is invariant.) The very same term tells us that the D3-brane turns on a monopole field configuration when it ends on the NS5-brane. What happens for a D4 ending on a NS5? It is difficult to answer from first principles, since the NS5-brane in type IIA cannot be studied perturbatively. The proposal is that the D4-brane is a source for the scalar $x^6$ of the multiplet of the NS5-brane: $x^6$ depends on $v$ . In other words, the NS5-brane can bend due to the presence of the D4-brane ending on it.

$S$ 规则以及 D6 膜穿过 NS5 膜会产生 D4 膜的结论仍然成立。但它和 IIB 的构造存在显著差异: 在 IIB 弦论中，我们知道负责 D3 膜端点终止在 NS5 上的项是 $\int F \wedge A^{(4)}$ 。(可以从 D3 膜终止在 D5 膜的情况计算得到该项，然后在 $A^{(4)}$ 不变的 S 对偶下推导出来。) 同一个项告诉我们，当 D3 膜终止在 NS5 膜上时会激发一个单极场构型。那 D4 膜终止在 NS5 膜上会发生什么？从第一性原理出发很难回答这个问题，因为 IIA 中的 NS5 膜无法微扰研究。目前的猜想是，D4 膜是 NS5 膜多重态中标量 $x^6$ 的源: $x^6$ 依赖于 $v$ 。换句话说，NS5 膜会因终止在其上的 D4 膜发生弯曲。

It is easy to see that this makes sense. The equation for $x^6$ is the Laplace equation

很容易看出这是自洽的: $x^6$ 满足的方程是拉普拉斯方程

$$\nabla^2 x^6 = \sum_{k=1}^{N_c} \delta(v - a_i), \tag{93}$$

with solution

解为

$$x^6 = \sum_{k=1}^{N_c} \log|v - a_k| + \text{ constant}, \tag{94}$$

where $a_i$ denote the positions of the D4-branes in (45). This is in agreement with the field theory: the coupling constant of the D4-branes is determined by the difference $\Delta x^6$ between the positions of two NS5-branes. As can be seen from (94), this depends logarithmically with $v$ , which is a 1-loop effect. Recall also that the bare coupling $g_{\text{cl}}$ in four dimensions is not a physical parameter (the exceptions are the superconformal theories with zero beta-functions, which we will discuss later); by dimensional transmutation, it is replaced by a mass scale $\Lambda$ . One can talk about the effective coupling that goes like $\tau(v) \sim \log(v/\Lambda)$ , exactly as described in (94). (In order to avoid cluttered notation, we set $\Lambda = 1$ in the subsequent discussion.) In fact, from the brane perspective, $g_{\text{cl}}$ could be identified only as a limiting value of $\Delta x^6$ . One could consider $x^6(v \to \infty)$ ; we see from (94) that this, however, does not tend to a constant but diverges, and so $g_{\text{cl}}$ cannot be defined. The only thing that makes sense is a constant that can be identified with $\Lambda$ .

其中 $a_i$ 表示 D4 膜在 (45) 方向的位置。这与场论结论一致:D4 膜的耦合常数由两个 NS5 膜的位置差 $\Delta x^6$ 决定。从 (94) 式可以看出，该耦合常数对 $v$ 呈对数依赖，这是 1 圈效应。我们还记得四维下的裸耦合 $g_{\text{cl}}$ 并非物理参数，零 β 函数的超共形理论除外，我们会在后文讨论；通过维度转变，它被质量标度 $\Lambda$ 取代。我们可以讨论按 $\tau(v) \sim \log(v/\Lambda)$ 变化的有效耦合，这正是 (94) 式所描述的。为避免符号杂乱，后续讨论中我们设 $\Lambda = 1$ 。实际上，从膜的视角看，$g_{\text{cl}}$ 只能被认作 $\Delta x^6$ 的极限值。我们可以考虑 $x^6(v \to \infty)$ ；但从 (94) 式可知，它不会趋近于常数，反而会发散，因此 $g_{\text{cl}}$ 无法被良定义。唯一有意义的是可对应为 $\Lambda$ 的常数。

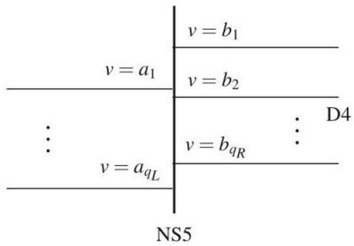
This should be compared with the case of three-dimensional gauge theories. The corresponding Laplace equation $\nabla^2 x^6 = \delta(\mathbf{x})$ , with $\mathbf{x}$ a vector in the three transverse directions (345), has the following solution: $x^6(\mathbf{x}) = x_0^6 + \sum_i \frac{1}{\mathbf{x}-\mathbf{x}_i}$ . The value of $x^6$ far from the D3-branes is $x_0^6$ , and $\Delta x_0^6$ can be identified as $1/g_{\text{cl}}^2$ .

这可以和三维规范理论的情况做对比。对应的拉普拉斯方程 $\nabla^2 x^6 = \delta(\mathbf{x})$，其中 $\mathbf{x}$ 是三个横向方向 (345) 中的一个矢量，有如下解: $x^6(\mathbf{x}) = x_0^6 + \sum_i \frac{1}{\mathbf{x}-\mathbf{x}_i}$。远离 D3 膜处 $x^6$ 的值为 $x_0^6$，且 $\Delta x_0^6$ 可对应为 $1/g_{\mathrm{cl}}^2$。

Let us now consider an NS5-brane, with $q_L$ D4-branes ending on the left at values $v = a_1, a_2, \ldots a_{q_L}$ and with $q_R$ D4-branes ending on the right at values $v = b_1, b_2, \ldots, b_{q_R}$ :

现在我们来考虑一个 NS5 膜，其中 $q_L$ 个 D4 膜左端终止于位置 $v = a_1, a_2, \ldots a_{q_L}$， $q_R$ 个 D4 膜右端终止于位置 $v = b_1, b_2, \ldots, b_{q_R}$：

(95)




The value of $x^6$ at large $v$ is

大 $v$ 下 $x^6$ 的值为

$$x^6 = \sum_{i=1}^{q_L} \log |v - a_i| - \sum_{j=1}^{q_R} \log |v - b_j| + \text{ const.} \tag{96}$$

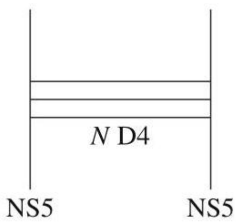
This has a well-defined limiting value for $v \to \infty$ if and only if $q_L = q_R$ .

当且仅当 $q_L = q_R$ 时，这对 $v \to \infty$ 存在良定义的极限值。

Let us consider first the $4d\mathcal{N} = 2\mathrm{U}(N)$ pure gauge theory, which can be realized from the following brane system:

我们首先来考虑 $4d\mathcal{N} = 2\mathrm{U}(N)$ 纯规范理论，它可以通过如下膜系统实现:

(97)



The first important effect is that the U(1) factor in the gauge symmetry is frozen. This can be seen as follows: Suppose that the D4-brane positions in the (45) directions are $a_i$ , with $i = 1, \ldots, N$ . The NS5-brane kinetic energy contains the term $\int d^4xd^2v \sum_{\mu=0}^{3} \partial_\mu x^6 \partial^\mu x^6$ . Using (96), we have

第一个重要效应是规范对称性中的 U(1) 因子被冻结了，我们可以按如下方式看出这一点: 假设 D4 膜在 (45) 方向的位置是 $a_i$ ，满足 $i = 1, \ldots, N$ 。NS5 膜的动能包含项 $\int d^4xd^2v \sum_{\mu=0}^{3} \partial_\mu x^6 \partial^\mu x^6$ 。利用 (96) 式，我们得到

$$k^2 \int d^4xd^2v \left| \mathrm{Re}\left( \sum_i \frac{\partial_\mu a_i}{v - a_i} \right) \right|^2 . \tag{98}$$

The integral over $v$ converges if and only if $\partial_\mu \left( \sum_i a_i \right) = 0$ . This implies $\sum_i a_i =$ constant, which can be set to zero by choosing the origin of the $v$ -plane appropriately. The gauge group is thus, in fact, $\mathrm{SU}(N)$ . Note that this is typical for four-dimensional gauge theories where $x^6$ behaves logarithmically.

对 $v$ 的积分当且仅当 $\partial_\mu \left( \sum_i a_i \right) = 0$ 时收敛。这说明 $\sum_i a_i =$ 是常数，通过适当选取 $v$ 平面的原点可将其设为零。因此规范群实际上是 $\mathrm{SU}(N)$ 。请注意，这在四维规范理论中是典型情况，其中 $x^6$ 表现出对数行为。

From the perspective of the 4 d$\mathcal{N} = 2$ theory, $x^6$ is in fact a real part of the holomorphic coupling $\tau$ , which can be regarded as a background field in a vector multiplet. The imaginary part of $\tau$ is a scalar field that propagates on the NS5- brane. The worldvolume theory of the latter is the 6 d$\mathcal{N} = (2,0)$ theory containing a tensor multiplet consisting of a self-dual two-form field and five real scalars. In fact, to specify the position of an NS5-brane in ten dimensions, we need only four scalars. The fact that there are five scalars can be understood from viewing type IIA theory on $\mathbb{R}^{10}$ as M theory on $\mathbb{R}^{10} \times S^1$ . Denoting the coordinates of $\mathbb{R}^{10}$ by $x^0, \ldots, x^9$ and that of the 11th dimension by $x^{10}$ , we see that the aforementioned fifth scalar field corresponds to the position of the NS5 in $x^{10}$ , which is a periodic variable with period $2\pi R$ , where $R$ is the radius of $S^1$ . Setting $s = \left(x^6 + ix^{10}\right)/R$ , equation (94) becomes

从 4 d$\mathcal{N} = 2$ 理论的视角来看，$x^6$ 实际上是全纯耦合 $\tau$ 的实部，可被视作向量多重态中的背景场。$\tau$ 的虚部是在 NS5 膜上传播的标量场。NS5 膜的世界体积理论是包含张量多重态的 6 d$\mathcal{N} = (2,0)$ 理论，该张量多重态由一个自对偶二形式场和五个实标量构成。实际上，要确定十维中 NS5 膜的位置，我们仅需要四个标量。存在五个标量可以这样理解:IIA 型理论在 $\mathbb{R}^{10}$ 上等价于 M 理论在 $\mathbb{R}^{10} \times S^1$ 上。将 $\mathbb{R}^{10}$ 的坐标记为 $x^0, \ldots, x^9$ ，第 11 维的坐标记为 $x^{10}$ ，我们可以发现上述第五个标量场对应 NS5 在 $x^{10}$ 中的位置，它是一个周期为 $2\pi R$ 的周期变量，其中 $R$ 是 $S^1$ 的半径。取 $s = \left(x^6 + ix^{10}\right)/R$ 时，式 (94) 变为

$$s = \sum_{i=1}^{N} \log\left(v - a_i\right) + \text{ const.} \tag{99}$$

The conventional definition for the holomorphic coupling in this theory is

该理论中全纯耦合的常规定义是

$$-i\tau(v) = s_2 - s_1 = \sum_{i=1}^{N} \log(v - a_i) - \left(-\sum_{i=1}^{N} \log(v - a_i)\right) = 2\sum_{i=1}^{N} \log(v - a_i)\,,$$

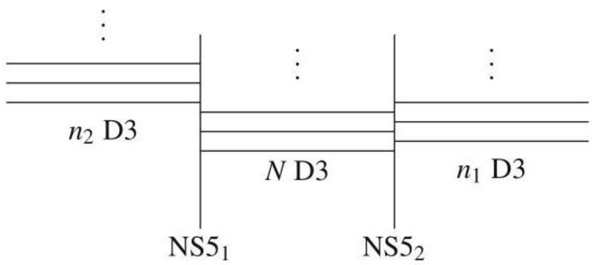
(100)

where $s_2$ and $s_1$ refer to the two NS5-branes and we put a plus (resp. minus) sign in the square brackets when the D3-branes are on the left (resp. right) of each NS5- brane. Here, $x^{10}$ becomes the theta angle.

其中 $s_2$ 和 $s_1$ 对应两个 NS5 膜，当 D3 膜位于每个 NS5 膜的左侧 (右侧) 时，我们在方括号中取正号 (负号)。此处 $x^{10}$ 即为 θ 角。

Let consider now a more general system:

现在我们考虑一个更一般的系统:




(101)

Here, the number of flavors under the SU $(N)$ gauge group is $N_f = n_1 + n_2$ . The holomorphic gauge coupling $\tau(v)$ is given by

此处，SU $(N)$ 规范群下的味数为 $N_f = n_1 + n_2$ 。全纯规范耦合 $\tau(v)$ 由下式给出

$$-i\tau(v) = s_2 - s_1 = \left[\sum_{i=1}^{N} \log(v - a_i) - \sum_{i=1}^{n_1} \log(v - m_i)\right]$$

(102)

$$-\left[-\sum_{i=1}^{N} \log(v - a_i) + \sum_{i=1}^{n_2} \log(v - m_i)\right].$$

Using the definition $N_f = n_1 + n_2$ , we can simplify the above expression as

利用定义 $N_f = n_1 + n_2$ ，我们可将上述表达式简化为

$$-i\tau(v) = 2\sum_{i=1}^{N} \log(v - a_i) - \sum_{i=1}^{N_f} \log(v - m_i)\,. \tag{103}$$

For large $v$ , we have

对于大的 $v$ ，我们有

$$-i\tau(v) \sim \left(2N - N_f\right)\log v \tag{104}$$

The coefficient of $\log v$ is equal to the one-loop beta-function, $2N - N_f$ , in agreement with the field theory.

$\log v$ 的系数等于一圈 β 函数 $2N - N_f$ ，这与场论结论一致。

Analogous to mirror symmetry in three dimensions that involves $S$ -duality (i.e., going from weak to strong coupling), we could obtain the information about four-dimensional theories at strong coupling by going from type IIA to M-theory. In doing so, one opens up a new direction $x^{10}$ . The NS5-brane is lifted to an M5-brane that is a point $\theta$ in the direction (10). The D4-brane is lifted to an M5-brane wrapping the (10) direction. We see that D4- and NS5-branes become the same object in M-theory, namely, an M5-brane. Since they touch each other, they become a single M5-brane with a same surface in 11 dimensions. The (0123) directions that are common between D4- and NS5-branes remain a flat $\mathbb{R}^4$ in M-theory, whereas the other two directions describe a complex curve, which must be holomorphic to preserve supersymmetry. Hence, the M-theory realization of this system is an M5-brane wrapping $\mathbb{R}^4 \times \sum$ where $\sum$ is a Riemann surface in the directions $(4,5,6,10)$ , since the NS5-brane fills the directions $(4,5)$ and the uplifted D4- brane fills the directions $(6,10)$ . The directions $(7,8,9)$ do not participate in this M-theory realization.

类似于三维中涉及 $S$ 对偶 (即从弱耦合到强耦合) 的镜像对称，我们可以通过从 IIA 型弦论跃迁到 M 理论，来获取四维强耦合理论的信息。在这个过程中，人们开启了一个新方向 $x^{10}$ 。NS5 膜提升为 M5 膜，它在方向 (10) 中是一个点 $\theta$ 。D4 膜提升为缠绕方向 (10) 的 M5 膜。我们可以看到，D4 膜与 NS5 膜在 M 理论中成为同一种对象，即 M5 膜。由于它们相互接触，最终会合并为十一维中具有相同曲面的单个 M5 膜。D4 膜与 NS5 膜共有的 (0123) 方向在 M 理论中仍是平坦的 $\mathbb{R}^4$ ，另外两个方向则描述一条复曲线，该曲线必须是全纯的才能保留超对称。因此，该系统的 M 理论实现是 M5 膜缠绕 $\mathbb{R}^4 \times \sum$，其中 $\sum$ 是方向 $(4,5,6,10)$ 中的黎曼曲面——这是因为 NS5 膜填充方向 $(4,5)$ ，而提升后的 D4 膜填充方向 $(6,10)$ 。方向 $(7,8,9)$ 不参与这个 M 理论实现。

On the NS5-brane, there is a tensor multiplet. If it is compactified on $\sum$ , we obtain the aforementioned gauge theory. Indeed, if $\sum$ has nontrivial one cycle, the tensor field reduces to a gauge field. For example, on a torus $T^2$ , the tensor field $T_{\mu\nu}$ reduces to two gauge fields, $A_{\mu a}$ and $A_{\mu b}$ , where $a$ and $b$ are the indices associated with the directions of each one cycle of the torus. However, the tensor multiplet is self-dual in six dimensions, $H = *H$ , where $H = dT$ . This implies that $A_{\mu a}$ and $A_{\mu b}$ are dual fields, since

NS5 膜上存在一个张量多重态。若将其在 $\sum$ 上紧化，我们就得到了上述规范理论。事实上，如果 $\sum$ 具有非平凡的一环，张量场就会约化为规范场。例如，在环面 $T^2$ 上，张量场 $T_{\mu\nu}$ 约化为两个规范场 $A_{\mu a}$ 和 $A_{\mu b}$ ，其中 $a$ 和 $b$ 是与环面每个一环的方向关联的指标。然而，该张量多重态在六维 $H = *H$ 中是自对偶的，满足 $H = dT$ 。这意味着 $A_{\mu a}$ 和 $A_{\mu b}$ 是对偶场，因为

$$F^a_{\mu\nu} = H^a_{\mu\nu} = \varepsilon^a_{\mu\nu\tau\rho b} H^{\tau\rho b} = \varepsilon_{\mu\nu\tau\rho} F^{\tau\rho b}. \tag{105}$$

Therefore, since the tensor is self-dual, every cycle in $\sum$ gives rise to an abelian gauge field in 4 d . This means that $\sum$ must have genus $N-1$ to reproduce the $\mathrm{SU}(N)$ gauge theory in question.

因此，由于张量是自对偶的，$\sum$ 中的每个一环都会在 4 d 中产生一个阿贝尔规范场。这说明 $\sum$ 必须具有亏格 $N-1$ ，才能重构我们讨论的 SU $(N)$ 规范理论。

## Application: Seiberg-Witten Curves from Branes

## 应用: 从膜构造 Seiberg-Witten 曲线

In type IIA string theory, the intersection between D4-brane and NS5-brane is singular. The approach of probing the space-time allows to calculate the 1-loop term in field theory, but not the instanton corrections which give rise to a smooth metric. In M-theory, the system is smooth, and we can expect the complete solution of the system, rather than just a semiclassical approximation [46].

在 IIA 型弦论中，D4 膜与 NS5 膜的相交是奇异的。探测时空的方法可以计算场论中的一圈项，但无法计算带来光滑度量的瞬子修正。在 M 理论中，该系统是光滑的，我们可以得到系统的完全解，而非仅仅是半经典近似 [46]。

Our goal is to determine the Riemann surface $\sum$ . We take

我们的目标是确定黎曼曲面 $\sum$ 。我们取

$$v = x^4 + ix^5,\ t = e^{-s} = \exp\left(-\frac{x^6 + ix^{10}}{R}\right). \tag{106}$$

Observe that $t$ is single-valued and so it is a good variable. We aim to write down the complex equation $F(t, v) = 0$ , with the following conditions:

注意到 $t$ 是单值的，因此它是一个合适的变量。我们的目标是写下复方程 $F(t, v) = 0$ ，满足如下条件:

- For a fixed $v, F(t, v)$ is a function of $t$ , and the roots of $F$ are the positions of the NS5-branes (at the given value of $v$ ). Therefore, the degree of $F$ as a polynomial in $t$ is the number of NS5-branes. For example, if we have two NS5-branes, then $F$ is quadratic in $t$ .

- 固定 $v, F(t, v)$ 时，$t$ 是 $F$ 的函数，$F$ 的根就是 NS5 膜在给定 $v$ 取值下的位置。因此 $F$ 作为 $t$ 多项式的次数等于 NS5 膜的数量。例如，如果我们有两个 NS5 膜，那么 $F$ 就是 $t$ 的二次多项式。

- For a fixed $t$ , the degree of $F$ as a polynomial in $v$ is the number $N$ of D4-branes suspended between the two NS5-branes.

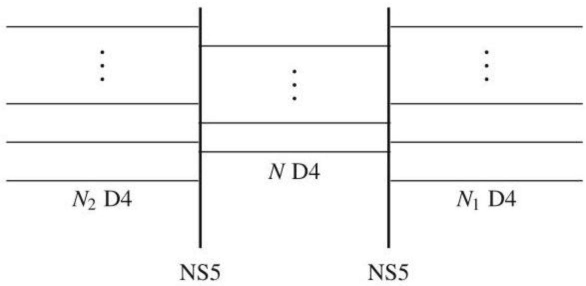
- 固定 $t$ 时，$F$ 作为 $v$ 多项式的次数等于悬挂在两个 NS5 膜之间的 D4 膜的数量 $N$ 。

Let us consider the following type IIA construction for the $SU(N)$ gauge theory with $N_f$ flavors:

下面我们考虑带 $N_f$ 味的 $SU(N)$ 规范理论的如下 IIA 构造:

(107)

$$N_f = N_1 + N_2$$




We can write down a complex curve of the form

我们可以写下如下形式的复曲线:

$$F(t,v) = A(v)\,t^2 + B(v)\,t + C(v) = 0, \tag{108}$$

where $A, B$ , and $C$ are polynomial in $v$ of degree $N_1, N$ , and $N_2$ , respectively.

其中 $A, B$ 和 $C$ 分别是 $v$ 的次数为 $N_1, N$ 和 $N_2$ 的多项式。

- At a zero of $C(v)$ , one of the roots (regarded as an equation in $t$ ) goes to $t = 0$ , corresponding to $x^6 = \infty$ . The existence of a root associated with $x^6 = \infty$ at a fixed $v$ (where $C(v) = 0$ ) corresponds to a semi-infinite D4-brane to the right of all NS5-branes.

- 在 $C(v)$ 的零点处,(视为 $t$ 中方程的) 其中一个根趋向 $t = 0$ , 对应 $x^6 = \infty$ 。在固定 $v$ (满足 $C(v) = 0$ ) 处存在一个对应 $x^6 = \infty$ 的根,意味着所有 NS5 膜的右侧存在一根半无限 D4 膜。

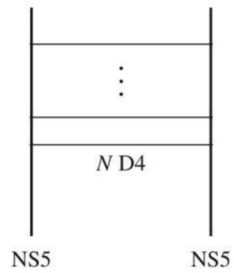
- Similarly, at a zero of $A(v)$ , a root of $F$ goes to $t = \infty$ , i.e., $x^6 = -\infty$ . This corresponds to a semi-infinite D4-branes on the left.

- 类似地,在 $A(v)$ 的零点处,$F$ 的一个根趋向 $t = \infty$ , 即 $x^6 = -\infty$ , 对应左侧存在一根半无限 D4 膜。

Consider first the case of the $\mathcal{N} = 2$ pure $\mathrm{SU}(N)$ gauge theory (i.e., $N_f = 0$ ), whose brane configuration is

首先考虑 $\mathcal{N} = 2$ 纯 $\mathrm{SU}(N)$ 规范理论的情况 (即 $N_f = 0$ ),其膜构型为

(109)




Since there is no semi-infinite D4-brane, there are no zeros of $A$ and $C$ . We conclude that $A$ and $C$ must be constants. After a rescaling of $t$ , we have

由于不存在半无限 D4 膜，因此 $A$ 和 $C$ 都没有零点。我们可以推出 $A$ 和 $C$ 必为常数。对 $t$ 做重标度后，我们得到

$$t^2 + B(v)\,t + 1 = 0. \tag{110}$$

Writing

记

$$\tilde{t} = t + \frac{1}{2}B \tag{111}$$

we have

可得

$$\tilde{t}^2 = \frac{1}{4}B(v)^2 - 1 \tag{112}$$

By rescaling and shifting $v$ , we see that $B$ can be put into the form

对 $v$ 做重标度和平移后，我们可以看出 $B$ 可以改写为如下形式

$$B(v) = v^N + u_2 v^{N-2} + u_3 v^{N-3} + \ldots + u_N. \tag{113}$$

where we have shifted $v$ to remove the $v^{N-1}$ term. This is the standard Seiberg-Witten curve for the SU $(N)$ pure gauge theory. This curve is in agreement with that discussed in [47, 48] , where the latter was written as

其中我们通过平移 $v$ 消去了 $v^{N-1}$ 项。这就是 SU $(N)$ 纯规范理论的标准西伯格-维滕曲线。该曲线与 [47, 48] 中讨论的结果一致，后者的表达式写为

$$y^2 = \left(x^N - \sum_{i=2}^{N} c_i x^{N-i}\right)^2 - \Lambda^{2N}. \tag{114}$$

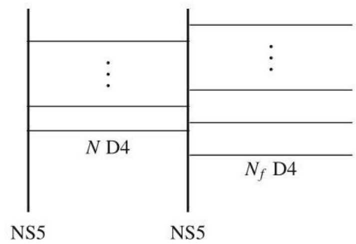
The coefficients $u_2, \dots, u_N$ are identified with the invariants $\mathrm{tr}\,\phi^i$ of the gauge theory. Note that the energy scale $\Lambda$ is set to one by rescaling $t = e^{i\tau}$ . From (110), for $t$ large, we see that $t \sim c \cdot v^N$ . For $t$ small, we have $t \sim c' \cdot v^{-N}$ . These behaviors represent the "bending" of the two NS5-branes as a result of being pulled on by D4- branes. (For $x^6 \to \pm\infty$ , the roots of $F(t, v)$ , as a function of $v$ for fixed $t$ , are at very large $|v|$ . These roots do not correspond, intuitively, to positions of D4-branes but are points "near infinity" on the bent NS5-branes.)

系数 $u_2, \dots, u_N$ 与规范理论的不变量 $\mathrm{tr}\,\phi^i$ 等同。注意，我们通过对 $t = e^{i\tau}$ 做重标度，将能标 $\Lambda$ 设为 1。由式 (110) 可知，当 $t$ 很大时，我们得到 $t \sim c \cdot v^N$；当 $t$ 很小时，我们得到 $t \sim c' \cdot v^{-N}$ 。这些行为体现了两张 NS5 膜被 D4 膜拉扯后产生的“弯曲”。(对于 $x^6 \to \pm\infty$ ，固定 $t$ 时，$F(t, v)$ 的根作为 $v$ 的函数位于极大的 $|v|$ 处。直观来看，这些根并不对应 D4 膜的位置，而是弯曲的 NS5 膜上“靠近无穷远”的点。)

In order to add $N_f$ flavors of hypermultiplets to the pure gauge theory, we put $N_f$ semi-infinite D4-branes, for example, on the right as in the following diagram:

为了给纯规范理论添加 $N_f$ 个超多重子味，我们引入 $N_f$ 张半无限 D4 膜，例如放在右侧，如下图所示:

(115)




We can take $A = 1$ and

我们可以取 $A = 1$ 和

$$C(v) = f \prod_{j=1}^{N_f} \left(v - m_j\right) \tag{116}$$

where $m_j$ , being the zeros of $C$ , are the position of the semi-infinite D4-branes which are the hypermultiplet bare masses and $f$ is a complex constant. The curve is

其中 $m_j$ 是 $C$ 的零点，对应半无限 D4 膜的位置，也就是超多重子的裸质量，且 $f$ 是一个复常数。曲线表达式为

$$\tilde{t}^2 = \frac{1}{4} B(v)^2 - f \prod_{j=1}^{N_f} \left(v - m_j\right) \tag{117}$$

where $\tilde{t} = t + B(v)/2$ , and we set now

其中 $\tilde{t} = t + B(v)/2$，我们现在设

$$B(v) = e\left(v^N + u_2 v^{N-2} + u_3 v^{N-3} + \ldots + u_N\right) \tag{118}$$

with $e$ and $u_i$ constants. We have shifted $v$ to remove the $v^{N-1}$ term. Of course, shifting $v$ by a constant to eliminate the $v^{N-1}$ term in $B$ will shift the $m_j$ by a constant. If $N_f \neq 2N$ , we can rescale $\tilde{t}$ and $v$ to set $e = f = 1$ . (Explicitly, this shift is $v \to v e^{\frac{2}{N_f - 2N}} f^{-\frac{1}{N_f - 2N}}$ and $\tilde{t} \to \tilde{t} e^{\frac{N_f}{N_f - 2N}} f^{-\frac{N}{N_f - 2N}}$ .) This is in agreement with the curves computed in [49, 50] .

其中 $e$ 和 $u_i$ 为常数。我们平移了 $v$ 以消除 $v^{N-1}$ 项。当然，将 $v$ 平移一个常数以消去 $B$ 中的 $v^{N-1}$ 项，会使 $m_j$ 随之平移一个常数。若满足 $N_f \neq 2N$，我们可以对 $\tilde{t}$ 和 $v$ 重新标度以令 $e = f = 1$。(具体而言，该平移为 $v \to v e^{\frac{2}{N_f - 2N}} f^{-\frac{1}{N_f - 2N}}$ 和 $\tilde{t} \to \tilde{t} e^{\frac{N_f}{N_f - 2N}} f^{-\frac{N}{N_f - 2N}}$。) 这与 [49, 50] 中计算得到的曲线一致。

The case of $N_f = 2N$ is special: the beta-function vanishes. We thus expect the (exactly marginal) coupling constant to be a good parameter of the theory. We can see this as follows. By rescaling $\tilde{t}$ and $v$ , it is possible to remove only one combination of $e$ and $f$ . (For example, the rescalings $v \to v f^{\frac{1}{N_f}} e^{-\frac{2}{N_f}}$ and $\tilde{t} \to \tilde{t} f e^{-1}$ send $e^2 \to e^2 f^{-1}$ and $f \to 1$ .) The asymptotic behavior as $v \to \infty$ for $N_f = 2N$ is

$N_f = 2N$ 的情况很特殊:β 函数为零。因此我们预期，(恰好临界的) 耦合常数是该理论一个良好的参数。我们可以如下理解: 通过对 $\tilde{t}$ 和 $v$ 做重标度，我们只能消去 $e$ 和 $f$ 的一个组合。(例如，重标度变换 $v \to v f^{\frac{1}{N_f}} e^{-\frac{2}{N_f}}$ 和 $\tilde{t} \to \tilde{t} f e^{-1}$ 给出 $e^2 \to e^2 f^{-1}$ 和 $f \to 1$。) 对于 $N_f = 2N$，当 $v \to \infty$ 时的渐近行为为

$$t^2 + e\left(v^N + \ldots\right) t + f\left(v^{2N} + \ldots\right) = 0. \tag{119}$$

Writing $y = t v^{-N}$ , we can rewrite this equation as

写下 $y = t v^{-N}$，我们可将该方程改写为

$$y^2 + e y + f = 0. \tag{120}$$

Hence, the asymptotic behavior of the curve is

因此，曲线的渐近行为为

$$t \sim \lambda_\pm v^N \tag{121}$$

where $\lambda_\pm$ are the two roots of (120). This means that the two NS5-branes are parallel at infinity - on both branches $t \sim v^N$ for $v \to \infty$ . The distance between them has a limit at infinity, which determines the gauge coupling. We can alternatively describe the same theory by putting $N$ semi-infinite D4-branes on the left and $N$ semi-infinite D4-branes on the right, as in Figure (107). The curve would read

其中 $\lambda_\pm$ 是式 (120) 的两个根。这说明两个 NS5 膜在无穷远处是平行的——在两个分支 $t \sim v^N$ 上对 $v \to \infty$ 都成立。它们在无穷远处的间距是有限值，这个间距决定了规范耦合常数。我们也可以采用另一种方式描述同一个理论: 左侧放置 $N$ 半无限 D4 膜，右侧放置 $N$ 半无限 D4 膜，如图 (107) 所示，此时曲线可以写为

$$A(v)\,t^2 + B(v)\,t + C(v) = 0, \tag{122}$$

where $A, B$ , and $C$ are now polynomials in $v$ of degree $N$ , and would describe the very same physics. The asymptotic behavior of the curve for large $v$ is $v^N\left(at^2 + bt + c\right) = 0$ , where $a, b$ , and $c$ are constants and the NS-branes are again parallel but not bended, sitting at given positions $t \sim \lambda_\pm$ , whose difference determines the gauge coupling. We can also see the Seiberg-Witten curve

其中 $A, B$ 和 $C$ 现在是关于 $v$ 的 $N$ 次多项式，描述的物理完全相同。当 $v$ 很大时，曲线的渐近行为是 $v^N\left(at^2 + bt + c\right) = 0$ ，其中 $a, b$ 和 $c$ 是常数，此时 NS 膜依旧平行且不再弯曲，固定在位置 $t \sim \lambda_\pm$ ，位置差决定了规范耦合常数。我们也可以将西伯格-维滕曲线看作

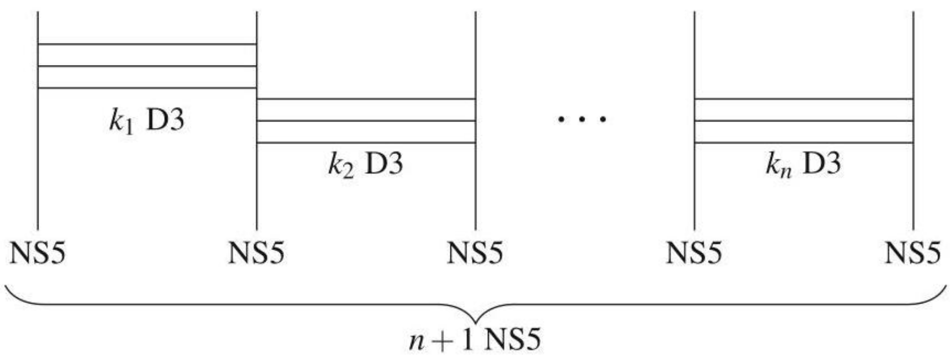
$$A(v)\,t^2 + B(v)\,t + C(v) = \prod_{i=1}^{N}\left(v - v_i(t)\right) = 0, \tag{123}$$

as an $N$ -fold covering of a sphere (the $t$ -plane) with four distinguished points (punctures) corresponding to the asymptotic positions of the two NS-branes and $t = 0, \infty$ which can be associated with the two sets of semi-infinite D4-branes. When all masses are zero, the theory is superconformal and a particular case of the class S theories (corresponding to a sphere with four punctures) that we will discuss in the next section.

球 (即 $t$ 平面) 的 $N$ 重覆盖，带有四个特殊点 (穿孔)，分别对应两个 NS 膜的渐近位置，以及与两组半无限 D4 膜对应的 $t = 0, \infty$ 。当所有质量都为零时，该理论是超共形的，属于 S 类理论的一个特例 (对应四个穿孔的球面)，我们将在下一节讨论。

The above configuration can be generalized in many ways. Let us look at the following example:

上述构造可以通过多种方式推广，来看下面的例子:




(124)

The gauge group is $\prod_{\alpha=1}^{n} \mathrm{SU}(k_\alpha)$ , and there are hypermultiplets transforming in the following representation of $\mathrm{SU}(k_1) \times \mathrm{SU}(k_2) \ldots \times \mathrm{SU}(k_n)$ :

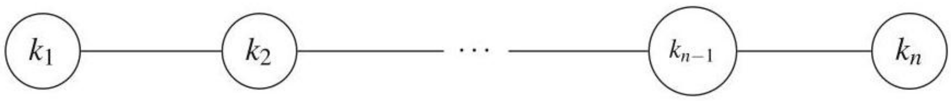
规范群为 $\prod_{\alpha=1}^{n} \mathrm{SU}(k_\alpha)$ ，且存在超多重态，按照 $\mathrm{SU}(k_1) \times \mathrm{SU}(k_2) \ldots \times \mathrm{SU}(k_n)$ 的下述表示变换:

$$\left(\mathbf{k}_1, \overline{\mathbf{k}}_2\right) \oplus \left(\mathbf{k}_2, \overline{\mathbf{k}}_3\right) \oplus \cdots \oplus \left(\mathbf{k}_{n-1}, \overline{\mathbf{k}}_n\right). \tag{125}$$

This can in fact be represented as a quiver diagram as follows:

这实际上可以用如下箭图表示:

(126)




The $\mathrm{SU}(k_\alpha)$ gauge group has a one-loop beta-function coefficient $-2k_\alpha + k_{\alpha-1} + k_{\alpha+1}$ . The curve is described by $F(v,t) = 0$ , where, since there are $(n+1)$ NS5- branes, $F$ has degree $n+1$ in $t$ . It thus takes the general form:

$\mathrm{SU}(k_\alpha)$ 规范群的一圈 β 函数系数为 $-2k_\alpha + k_{\alpha-1} + k_{\alpha+1}$ ，曲线由 $F(v,t) = 0$ 描述，由于共有 $(n+1)$ 个 NS5 膜，因此 $F$ 关于 $t$ 是 $n+1$ 次的，其一般形式为:

$$0 = F(v,t) = t^{n+1} + f_1(v)\, t^n + f_2(v)\, t^{n-1} + \ldots + f_n(v)\, t + 1. \tag{127}$$

The coefficients of $t^{n+1}$ and $t^0$ are nonzero constant to ensure that there are no semi-infinite D4-branes; their coefficients can be set to 1 by rescaling $F$ and $t$ . Alternatively, we can write $F$ in terms of its roots:

$t^{n+1}$ 和 $t^0$ 的系数为非零常数，保证不存在半无限 D4 膜；通过对 $F$ 和 $t$ 重新标度，可以将它们的系数设为 1。我们也可以将 $F$ 用其根表示为:

$$F = \prod_{\alpha=0}^{n} \left(t - t_\alpha(v)\right). \tag{128}$$

If the position $t_\alpha(v)$ of the $\alpha$ -th NS5-brane (with $\alpha = 0, 1, \ldots, n$ ) for large $v$ is $t_\alpha(v) \sim v^{a_\alpha}$ , then from the relation of the beta-function to the bending of NS5- branes, we have

当 $\alpha = 0, 1, \ldots, n$ 成立时，若 $v$ 很大时第 $\alpha$ 个 NS5 膜的位置 $t_\alpha(v)$ 为 $t_\alpha(v) \sim v^{a_\alpha}$ ，那么根据 β 函数和 NS5 膜弯曲的关系，我们有

$$a_\alpha - a_{\alpha-1} = 2k_\alpha - k_{\alpha-1} - k_{\alpha+1},\ \alpha = 1, 2, \ldots, n. \tag{129}$$

Since the term $t^0$ in (127) is independent of $v$ , we have

由于式 (127) 中的项 $t^0$ 与 $v$ 无关，因此我们得到

$$\sum_{\alpha=0}^{n} a_\alpha = 0 \tag{130}$$

We thus have

因此我们有

$$a_\alpha = k_\alpha - k_{\alpha+1}. \tag{131}$$

where we take $k_0 = k_{n+1} = 0$ . Given $a_\alpha$ , one can determine the coefficients $f_\alpha(v)$ . The parameters that cannot be set to zero by rescaling or shifting $v$ are the moduli of the theory.

其中我们取 $k_0 = k_{n+1} = 0$ 。给定 $a_\alpha$ 就可以确定系数 $f_\alpha(v)$ 。无法通过重新标度或平移 $v$ 归零的参数就是该理论的模。

There is another way to obtain hypermultiplets, namely, using D6-branes. We know that lifting a D6-brane to M-theory gives rise to a Taub-NUT space in the directions $(4, 5, 6, 10)$ parametrized by the variables $v$ and $t$ . A Hanany-Witten brane configuration with D4-, NS5-, and D6-branes will uplift to an M5-brane wrapping a Riemann surface in a Taub-Nut geometry. One can check that this gives the same Seiberg-Witten curve as before [46].

还有另一种方法得到超多重子, 即利用 D6 膜。我们知道, 将 D6 膜提升到 M 理论会在方向 $(4, 5, 6, 10)$ 上产生一个陶布-纳特空间，该空间由变量 $v$ 和 $t$ 参数化。包含 D4 膜、NS5 膜和 D6 膜的汉纳尼-威滕膜构型会提升为 M5 膜，包裹陶布-纳特几何中的一个黎曼曲面。可以验证，由此得到的西伯格-维滕曲线与之前的结果一致 [46]。

## Four-Dimensional $\mathcal{N} = 2$ Theories of Class S

## 四维 $\mathcal{N} = 2$ S 类理论

In this section, we consider the 4 d$\mathcal{N} = 2$ superconformal field theories that arise from compactifying a 6d $\mathcal{N} = (2, 0)$ theory on a Riemann surface $\sum_g$ of genus $g$ . These are known as the theories of class $S$ [5,6]. We start our discussion by considering $\sum_g$ with no punctures. We will add punctures to $\sum_g$ later on.

本节中，我们讨论将 6 维 $\mathcal{N} = (2, 0)$ 理论紧致化在亏格为 $g$ 的黎曼曲面 $\sum_g$ 上得到的 4 d$\mathcal{N} = 2$ 超共形场论，这类理论被称为 S $S$ 类理论 [5,6]。我们从无 puncture 的 $\sum_g$ 开始讨论，后续再为 $\sum_g$ 引入 puncture。

The 6d $\mathcal{N} = (2, 0)$ theory [51] is the mysterious theory living on the worldvolume of $N$ coincident M5-branes. On a single M5-brane, there exists a self-dual tensor and five real scalars parametrizing the transverse positions of the brane. The theory living on $N$ coincident M5-branes is a non-abelian theory with tensor multiplets and no Lagrangian description, which reduces at low energy to a superconformal theory with 16 supercharges. For simplicity, let us take the non-abelian symmetry algebra of the 6 d$\mathcal{N} = (2,0)$ theory to be of type $A_{N-1}$ .

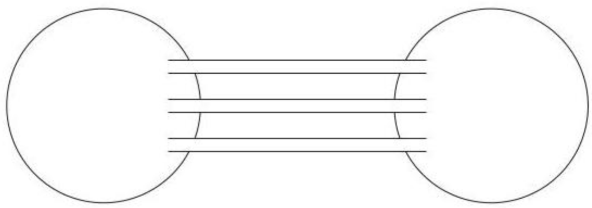
> 6 维 $\mathcal{N} = (2,0)$ 理论 [51] 是存在于 $N$ 个重合 M5 膜世界体上的神秘理论。单个 M5 膜上存在一个自对偶张量和五个实标量场，参数化描述膜的横向位置。$N$ 个重合 M5 膜上的理论是一个非阿贝尔理论，包含张量多重态且无拉格朗日描述，它在低能下约化为具有 16 个超荷的超共形理论。为简化起见，我们取 6 d$\mathcal{N} = (2,0)$ 理论的非阿贝尔对称代数为 $A_{N-1}$ 型。

If $g \neq 1$ , the curvature of $\sum_g$ generally breaks all supersymmetry. In order to preserve a certain amount of supersymmetry, we introduce a background $R$ - symmetry gauge field on $\sum_g$ that partially cancels the curvature. In particular, we consider a $\mathrm{Spin}(2) \times \mathrm{Spin}(3) \cong \mathrm{U}(1) \times \mathrm{SU}(2)$ subgroup of the $\mathrm{Spin}(5)\, R$ - symmetry of the $\mathcal{N} = (2,0)$ theory and identify the background gauge field of the $\mathrm{U}(1)$ symmetry as (minus) the spin connection of $\sum_g$ . In order to take the limit to a four-dimensional theory, we consider the limit in which the area of $\sum_g$ goes to zero. Indeed, the aforementioned $\mathrm{U}(1) \times \mathrm{SU}(2)$ symmetry should be identified with the $R$ -symmetry of the 4 d$\mathcal{N} = 2$ theory.

> 若 $g \neq 1$ ，$\sum_g$ 的曲率通常会破缺所有超对称。为了保留一定数量的超对称，我们在 $\sum_g$ 上引入背景 $R$ 对称规范场来部分抵消曲率。具体来说，我们考虑 $\mathcal{N} = (2,0)$ 理论的 $\mathrm{Spin}(5)\, R$ 对称的一个 $\mathrm{Spin}(2) \times \mathrm{Spin}(3) \cong \mathrm{U}(1) \times \mathrm{SU}(2)$ 子群，并将 $\mathrm{U}(1)$ 对称的背景规范场等同于 (负的) $\sum_g$ 的自旋联络。为了得到四维理论的极限，我们取 $\sum_g$ 的面积趋于零。事实上，上述 $\mathrm{U}(1) \times \mathrm{SU}(2)$ 对称应当等同于 4 d$\mathcal{N} = 2$ 理论的 $R$ 对称。

We can give an alternative interpretation of these theories as follows: We can decompose $\sum_g$ in such a way that it consists of $2(g-1)$ three-punctured spheres and $3(g-1)$ tubes connecting them. We depict the example of $g = 2$ below:

> 我们可以对这些理论给出另一种诠释: 我们可以将 $\sum_g$ 分解为 $2(g-1)$ 个三 puncture 球面和连接它们的 $3(g-1)$ 个管，$g = 2$ 的例子如下图所示:



(132)

We now take almost all the area of $\sum_g$ to be in the tubes. By dimensionally reducing on the circle, each of the tubes can be associated with a 5 d$\mathcal{N} = 2\mathrm{SU}(N)$ super Yang-Mills (SYM) theory on a segment. These 5 d$\mathcal{N} = 2$ SYM theories are coupled to the $4d\mathcal{N} = 2$ theories represented by two three-punctured spheres at the two ends of the segment, which provide two SU(N) flavor symmetries, one on the left and the other on the right. By taking appropriate boundary conditions for the scalar fields in the 5 d theory, each tube reduces to a 4 d$\mathcal{N} = 2\mathrm{SU}(N)$ vector multiplet in the 4 d limit. (These boundary conditions were discussed in [52]. Let us denote the five real scalars in the 5 d$\mathcal{N} = 2$ SYM by $\phi_i$ (with $i = 1, \ldots, 5$ ). They transform in the vector representation of $\mathrm{SO}(5)_R R$ -symmetry. We decompose them into $\phi_{a=1,2}$ and $\phi_{\alpha=1,2,3}$

in the vector representation of the SO(2) and SO(3) subgroups of Spin(5) $_R$ , respectively. We then impose a Neumann boundary condition $D_n \phi_{a=1,2}\ | = 0$ for the former and a generalized Dirichlet boundary condition $\phi_{\alpha=1,2,3}\ | = \mu_{\alpha=1,2,3}$ on the latter, where $\mu_{\alpha=1,2,3}$ are scalar operators of dimension 2 in the triplet of $\mathrm{SU}(2)_R R$ -symmetry of the 4 d$\mathcal{N}=2$ theory at the end of the segment. They are the bottom component of the flavor current multiplet associated with the flavor symmetry SU $(N)$ , known as the moment map operators. Suppose that a tube has a radius $R$ and a length $L$ . Reducing the 6d $\mathcal{N}=(2,0)\,A_{N-1}$ theory along the circular direction, we have the 5 d$\mathcal{N}=2$ SU $(N)$ SYM on a segment, with 5 d gauge coupling $1/g_5^2 \sim 1/R$ . We then take the limit $L \to 0$ in such a way that the 4 d gauge coupling $1/g_4^2 \sim L/R$ is fixed. The three real scalars $\phi_{\alpha=1,2,3}$ , are removed by the generalized Dirichlet boundary condition, and the two real scalars $\phi_{a=1,2}$ along with the gauge field become the 4 d $\mathcal{N}=2$ vector multiplet.)

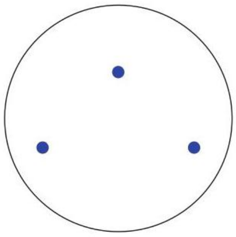
我们现在让 $\sum_g$ 几乎全部面积都位于管中。通过沿圆维度约化，每个管都可以对应线段上的一个 5 d$\mathcal{N}=2$SU $(N)$ 超杨-米尔斯 (SYM) 理论。这些 5 d$\mathcal{N}=2$ SYM 理论与线段两端两个三穿孔球面表示的 4d$\mathcal{N}=2$ 理论耦合，这两个端点各提供一个 SU(N) 味对称性，分别在左端和右端。对 5 d 理论中的标量场施加合适边界条件后，每个管在 4 d 极限下约化为一个 4 d$\mathcal{N}=2$SU $(N)$ 矢量多重态。(这些边界条件已在文献 [52] 中讨论。我们将 5 d$\mathcal{N}=2$ SYM 中的五个实标量记为 $\phi_i$ (满足 $i=1,\ldots,5$ )。它们按 $\mathrm{SO}(5)_R R$ 对称性的矢量表示变换。我们将它们分解为 Spin(5) $_R$ 的 SO(2) 和 SO(3) 子群矢量表示下的 $\phi_{a=1,2}$ 和 $\phi_{\alpha=1,2,3}$ ，分别对应两个子群。随后我们对前者施加诺伊曼边界条件 $D_n \phi_{a=1,2}\ | = 0$ ，对后者施加广义狄利克雷边界条件 $\phi_{\alpha=1,2,3}\ | = \mu_{\alpha=1,2,3}$ ，其中 $\mu_{\alpha=1,2,3}$ 是线段端点处 4 d$\mathcal{N}=2$ 理论的 $\mathrm{SU}(2)_R R$ 对称性三重态中维度为 2 的标量算符。它们是与味对称性 SU $(N)$ 关联的味流多重态的底分量，被称为矩映射算符。设一个管的半径为 $R$ ，长度为 $L$ 。将 6d $\mathcal{N}=(2,0)\,A_{N-1}$ 理论沿圆周方向约化，我们得到线段上的 5 d$\mathcal{N}=2$ SU $(N)$ SYM，其 5 d 规范耦合为 $1/g_5^2 \sim 1/R$ 。随后我们取极限 $L \to 0$ ，过程中保持 4 d 规范耦合 $1/g_4^2 \sim L/R$ 固定。三个实标量 $\phi_{\alpha=1,2,3}$ 被广义狄利克雷边界条件移除，两个实标量 $\phi_{a=1,2}$ 与规范场共同构成 4 d $\mathcal{N}=2$ 矢量多重态。)

Given the above construction, it makes sense to consider the 4 d$\mathcal{N}=2$ theory obtained by compactifying the 6d$\mathcal{N}=(2,0)$ theory on a three-punctured sphere

给定上述构造，研究通过将 6d$\mathcal{N}=(2,0)$ 理论紧致化在三穿孔球面上得到的 4 d$\mathcal{N}=2$ 理论是合理的。

(133)where each puncture, denoted by a blue dot, gives rise to an SU $(N)$ symmetry. This 4 d$\mathcal{N}=2$ theory is known as the $T_N$ theory (or trinion). It plays a crucial role as a basic building block in constructing other theories in this class. Each puncture arises from the boundary condition of the 5 d SYM at the end of a segment, and the type of puncture we are dealing with here is known as a full (or maximal) puncture. We will briefly mention other types of punctures later.

其中每个由蓝点标记的 puncture(穿孔) 都会产生一个 SU $(N)$ 对称性。该 4 d$\mathcal{N}=2$ 理论就是熟知的 $T_N$ 理论 (也称为三孔球面理论)。它作为基础构造块在构建该类其他理论中发挥关键作用。每个 puncture 来源于片段末端 5 d 超对称杨-米尔斯理论的边界条件，本文讨论的这类 puncture 被称为满 (极大)puncture。我们稍后会简要介绍其他类型的 puncture。



Let us mention that the $T_2$ theory is a theory of four free hypermultiplets, which in $\mathcal{N} = 1$ language consists of eight chiral multiplets (half-hypermultiplets) in the trifundamental representation of the $\mathrm{SU}(2)^3$ flavor symmetry. The latter can be denoted by $Q_{ai\alpha}$ , with $a, i, \alpha = 1, 2$ the indices of each SU (2) factor in $\mathrm{SU}(2)^3$ . The $T_3$ theory (is also known as the $E_6$ Minahan-Nemeschansky theory [53]), on the other hand, does not admit a 4 d$\mathcal{N} = 2$ Lagrangian description. (We remark, however, that there are 4 d$\mathcal{N} = 1$ Lagrangian theories that flow to the $T_3$ theory in the infrared; see [54-56].) It is a strongly coupled SCFT with an $E_6$ flavor symmetry, where the $\mathrm{SU}(2)^3$ symmetry from the punctures gets enhanced to $E_6$ . In general, the $T_N$ theory, with $N \geq 4$ , is a strongly coupled SCFT with $\mathrm{SU}(N)^3$ flavor symmetry. We will justify some of these statements later.

需要说明的是，$T_2$ 理论是四个自由超多重态的理论，用 $\mathcal{N} = 1$ 语言描述，它包含八个手征多重态 (半超多重态)，属于 $\mathrm{SU}(2)^3$ 味对称性的三基本表示。后者可以记为 $Q_{ai\alpha}$ ，其中 $a, i, \alpha = 1, 2$ 是 $\mathrm{SU}(2)^3$ 中每个 SU (2) 因子的指标。另一方面，$T_3$ 理论 (也称为 $E_6$ 米纳汉-内梅尚斯基理论 [53]) 不存在 4 d$\mathcal{N} = 2$ 拉格朗日描述。(但我们注意，确实存在 4 d$\mathcal{N} = 1$ 拉格朗日理论，其在红外极限流向 $T_3$ 理论；参见文献 [54-56]。) 它是具有 $E_6$ 味对称性的强耦合超共形场论，来自 puncture 的 $\mathrm{SU}(2)^3$ 对称性增强为 $E_6$ 。一般来说，当满足 $N \geq 4$ 时，$T_N$ 理论是具有 $\mathrm{SU}(N)^3$ 味对称性的强耦合超共形场论。我们稍后会对其中部分结论给出论证。

Let us go back to example (132). The corresponding 4 d theory is described by two copies of the $T_N$ theory coupled by three 4 d$\mathcal{N} = 2\mathrm{SU}\,(N)$ vector multiplets:

我们回到例子 (132)。对应的 4 d 理论由两份 $T_N$ 理论通过三个 4 d$\mathcal{N} = 2\mathrm{SU}\,(N)$ 矢量多重态耦合描述:

$$T_N\left(\mathrm{SU}(N)_1, \mathrm{SU}(N)_2, \mathrm{SU}(N)_3\right) \times T_N\left(\mathrm{SU}(N)_1, \mathrm{SU}(N)_2, \mathrm{SU}(N)_3\right) / \tau_{1,2,3}$$

$$\left(\mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \times \mathrm{SU}(N)_3\right).$$

(134)

where $\mathrm{SU}(N)_{1,2,3}$ denotes the SU $(N)$ symmetry arises from each puncture1,2,3, and the quotient denotes the gauging of $\prod_{i=1}^{3} \mathrm{SU}(N)_i$ flavor symmetry with gauge couplings $\tau_{1,2,3}$ . For $N = 2$ , this admits the conventional Lagrangian description as the SU (2) × Spin (4) gauge theory with two copies of half-hypermultiplets in the [2; 4] representation:

其中 $\mathrm{SU}(N)_{1,2,3}$ 表示源自每个 puncture 1、2、3 的 SU $(N)$ 对称性，商运算表示用规范耦合 $\tau_{1,2,3}$ 对 $\prod_{i=1}^{3} \mathrm{SU}(N)_i$ 味对称性做规范处理。当满足 $N = 2$ 时，该理论存在常规拉格朗日描述，对应具有两份 [2; 4] 表示半超多重态的 SU (2) × Spin (4) 规范理论:

$$(\mathrm{SU}(2)) \tag{135}$$

The equivalence of descriptions $(134)_{N=2}$ and (135) can be seen as follows: We remark that Spin (4) ≅ SU (2) × SU (2) and the vector representation 4 of Spin (4) is identified with [2; 2] of SU (2) × SU (2) . Indeed, the 16 half-hypermultiplets which are two copies of [2; 4] of SU (2) × Spin (4) in (135) can be regarded as those coming from two copies of the $T_2$ theory, each contains half-hypermultiplets in [2; 2; 2] of SU (2) × SU (2) × SU (2) , and gauging the SU (2)×Spin (4) symmetry in (135) is equivalent to gauging the SU (2)×SU (2)×SU (2) symmetry in (134).

描述 $(134)_{N=2}$ 与 (135) 的等价性可以如下理解: 我们注意到 Spin (4) ≅ SU (2) × SU (2) 和 Spin (4) 的向量表示 4 与 SU (2) × SU (2) 的 [2; 2] 等价。事实上，(135) 中 16 个半超多重子是 SU (2) × Spin (4) 的 [2; 4] 的两份拷贝，可将其视作来自两份 $T_2$ 理论的拷贝，每个拷贝都包含 SU (2) × SU (2) × SU (2) 的 [2; 2; 2] 表示下的半超多重子，而对 (135) 中 SU (2) × Spin (4) 对称性做规范作用等价于对 (134) 中 SU (2) × SU (2) × SU (2) 对称性做规范作用。

## Punctures and M-Theory Interpretation

## 缺陷与 M 理论诠释

Let us consider a 4d $\mathcal{N} = 2$ SCFT with a flavor symmetry SU $(N)$ . The bottom component of the flavor current multiplet contains the scalar operators $\boldsymbol{\mu} = (\mu_+, \mu_0, \mu_-)$ in the triplet of the $\mathrm{SU}(2)_R R$ -symmetry and has scaling dimension 2. Each component of $\boldsymbol{\mu}$ transforms in the adjoint representation of SU $(N)$ ; they are also known as the moment map operators. We can give a nilpotent VEV to $\mu_+$ which can be put into the Jordan normal form as

我们考虑一个具有味对称性 SU $(N)$ 的 4 维 $\mathcal{N} = 2$ 超共形场论 (SCFT)。味流多重态的底分量包含标量算子 $\boldsymbol{\mu} = (\mu_+, \mu_0, \mu_-)$ ，这些标量属于 $\mathrm{SU}(2)_R R$ 对称性的三重态，标度维度为 2。$\boldsymbol{\mu}$ 的每个分量都按 SU $(N)$ 的伴随表示变换，它们也被称为矩映射算子。我们可以给 $\mu_+$ 赋予一个幂零真空期望值 (VEV)，该真空期望值可以写成若尔当标准型:

$$\langle \mu_+ \rangle = J_{n_1} \oplus J_{n_2} \oplus \dots \tag{136}$$

where $N = \sum_i n_i$ and $J_n$ denotes a Jordan block of size $n \times n$ with zeros along the diagonal and $n - 1$ nonzero entries on the line above the diagonal. We will assume that $n_1 \geq n_2 \geq n_3 \geq \dots$ and denote by $Y$ the sequence $[n_1, n_2, n_3, \cdots]$ , which is a partition of $N$ . We will also use a superscript to denote a repetition of the entries, for example, the partition $4 = 2 + 1 + 1$ will be denoted by $\left[4, 1^2\right]$ .

其中 $N = \sum_i n_i$ 和 $J_n$ 表示大小为 $n \times n$ 的若尔当块，对角线上全为零，对角上方次对角线有 $n-1$ 个非零元。我们约定 $n_1 \geq n_2 \geq n_3 \geq \ldots$ ，并用 $Y$ 表示序列 $[n_1, n_2, n_3, \cdots]$ ，这是 $N$ 的一个分拆。我们用上标表示分拆项的重复，例如分拆 $4 = 2 + 1 + 1$ 记为 $\left[4, 1^2\right]$ 。

Let us now discuss theories of class $S$ associated with Riemann surfaces with punctures. We assign a partition $Y$ to each puncture of the Riemann surface. Starting from an $\mathrm{SU}(N)$ flavor symmetry, the assignment of the partition $Y$ to a puncture sets $\langle \mu_+ \rangle = J_Y$ to that puncture. As a result, this higgses the flavor symmetry $\mathrm{SU}(N)$ to a subgroup $G_Y$ . In particular, if $Y = \left[s_1^{r_1}, s_2^{r_2}, \ldots\right]$ where $r_i$ is the number of times that $s_i$ appears in $Y$ such that $s_1 > s_2 > \cdots$ , then $G_Y = \left(\prod_i \mathrm{U}(r_i)\right) / \mathrm{U}(1)$ . There are also the Nambu-Goldstone bosons and their superpartners associated with this higgsing of the flavor symmetry (see [57] for more details). This process is also known as a partial closure of the puncture.

现在我们来讨论与带缺陷黎曼面相联系的 S 类 $S$ 理论。我们给黎曼面的每个缺陷分配一个分拆 $Y$ 。从一个 $\mathrm{SU}(N)$ 味对称性出发，给缺陷分配分拆 $Y$ 就相当于给该缺陷赋予 $\langle \mu_+ \rangle = J_Y$ 。结果，这一操作将味对称性 $\mathrm{SU}(N)$ 希格斯化为子群 $G_Y$ 。特别地，若 $Y = \left[s_1^{r_1}, s_2^{r_2}, \ldots\right]$ ，其中 $r_i$ 是满足 $s_1 > s_2 > \cdots$ 的 $s_i$ 在 $Y$ 中出现的次数，则有 $G_Y = \left(\prod_i \mathrm{U}(r_i)\right) / \mathrm{U}(1)$ 。味对称性的希格斯化过程会产生南部-戈德斯通玻色子及其超对称伙伴 (更多细节参见文献 [57])，这一过程也称为缺陷的部分闭合。

For simplicity, we will focus on two types of partitions, namely, $Y = \left[1^N\right]$ and $Y = [N-1, 1]$ . We see clearly that the partition $Y = \left[1^N\right]$ corresponds to $\langle \mu_+ \rangle = 0$ , and so this is a trivial operation that does not involve any Higgsing. The corresponding flavor symmetry is therefore $\mathrm{SU}(N)$ . The puncture associated with $Y = \left[1^N\right]$ is known as the full (or maximal) puncture. This is the type of puncture we have in the $T_N$ theory. On the other hand, the partition $Y = [N-1, 1]$ has a U(1) flavor symmetry associated with it, and this is known as the simple (or minimal) puncture. The reader is referred to [57,58] for more details of other types of punctures. Note that for the special case of $N = 2$ (i.e., the theories of class $S$ coming from the $6d$ $(2,0)$ theory of the $A_1$ type), the full and simple punctures are the same, and each brings about an SU(2) flavor symmetry, and we will not distinguish them in the subsequent discussion.

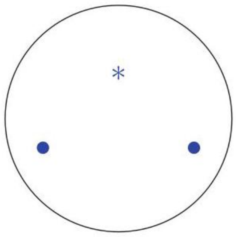
为简便起见，我们将聚焦两类分划，即 $Y = \left[1^N\right]$ 和 $Y = [N-1, 1]$ 。不难看出，分划 $Y = \left[1^N\right]$ 对应 $\langle \mu_+ \rangle = 0$ ，因此这是不涉及任何希格斯化的平凡操作，对应的味对称就是 $\mathrm{SU}(N)$ 。与 $Y = \left[1^N\right]$ 关联的 puncture 被称为满 (极大)puncture，这正是 $T_N$ 理论中存在的 puncture 类型。另一方面，分划 $Y = [N-1, 1]$ 带有 U(1) 味对称，对应的被称为单 (极小)puncture。其他类型 puncture 的更多细节读者可参考文献 [57,58]。注意对于 $N = 2$ 的特殊情形 (即来自 $A_1$ 型 $6d$ $(2,0)$ 理论的 $S$ 类理论)，满 puncture 和单 puncture 是等价的，二者都会带来 SU(2) 味对称，因此我们在后续讨论中不区分它们。

We can study, for example, the 6d $\mathcal{N} = (2,0)$ theory compactified on a sphere with two full punctures and a simple puncture, depicted by

例如，我们可以研究 6 维 $\mathcal{N} = (2,0)$ 理论紧化在带有两个满 puncture 和一个单 puncture 的球面上，如下图所示:

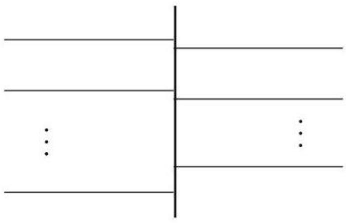
(138)where the number of semi-infinite D4-branes on each side of the NS5-branes is $N$ . The two full punctures are associated with the semi-infinite D4-branes on each side, and the simple puncture is associated with the NS5-branes. In the special case of $N = 2$ , this is indeed the $T_2$ theory, which is the theory of four free hypermultiplets, as expected.

其中 NS5 膜每一侧半无限 D4 膜的数量为 $N$ 。两个满 puncture 分别对应两侧的半无限 D4 膜，单 puncture 对应 NS5 膜。在 $N = 2$ 的特殊情形下，这确实就是 $T_2$ 理论，如预期所言该理论是四个自由超多重态的理论。



(137)where each dot denotes a full puncture and a star denotes a simple puncture. This theory is, in fact, a theory of $N^2$ free hypermultiplets, where in $\mathcal{N} = 1$ language the chiral multiplets can be written as $Q^i_a$ and $\widetilde{Q}^a_i$ where $a, i = 1, 2, \ldots, N$ are the indices associated with each $\mathrm{SU}(N)$ coming from the full puncture and $(Q, \widetilde{Q})$ carries charge $(+1, -1)$ under the $\mathrm{U}(1)$ symmetry coming from the simple puncture. In order to see this, we notice that (137) is in fact the M-theory realization of the following type IIA construction [46]:

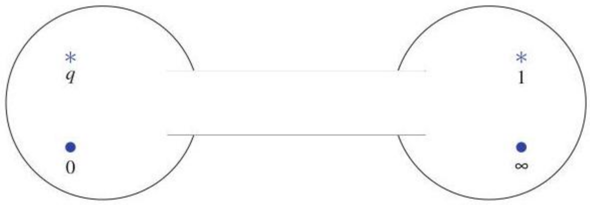
其中每个圆点代表满 puncture，星号代表单 puncture。该理论实际上是 $N^2$ 个自由超多重态的理论，在 $\mathcal{N} = 1$ 语言中，手征多重态可以写为 $Q^i_a$ 和 $\widetilde{Q}^a_i$ ，其中 $a, i = 1, 2, \ldots, N$ 是与满 puncture 带来的每个 $\mathrm{SU}(N)$ 关联的指标，而 $(Q, \widetilde{Q})$ 在单 puncture 带来的 $\mathrm{U}(1)$ 对称下带 $(+1, -1)$ 荷。为了说明这一点，我们注意到 (137) 实际上是如下 IIA 型构型的 M 理论实现 [46]:




Many other interesting SCFTs can be constructed by gluing such a Riemann sphere together. For example, we can take two copies of (137) and (weakly) gauge the $\mathrm{SU}(N)$ diagonal subgroup of the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ arising from a pair of full punctures:

我们可以通过将这种黎曼球粘合在一起构造许多其他有趣的超共形场论 (SCFT)。例如，我们可以取两份 (137)，并 (弱地) 规范来自一对满 puncture 的 $\mathrm{SU}(N) \times \mathrm{SU}(N)$ 的 $\mathrm{SU}(N)$ 对角子群:

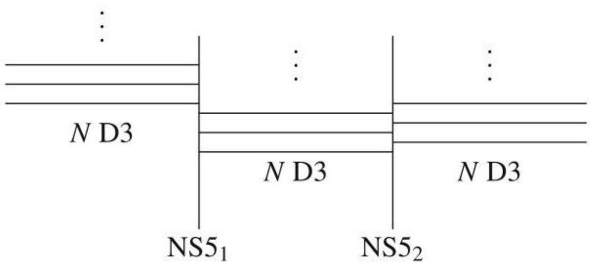
(139)




Indeed, it is the M-theory realization of the following type IIA system:

没错，它就是如下 IIA 系统的 M 理论实现:

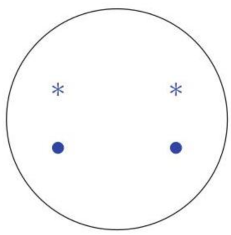
(140)




This realizes the SU $(N)$ gauge theory with $2N$ flavors. At a general value of the gauge coupling, this theory can be described by

这实现了带有 $2N$ 个味的 SU $(N)$ 规范理论。在规范耦合的一般取值下，该理论可以描述为

(141)



As discussed in section "Hanany-Witten Brane Configurations in Four Dimensions" (see (123)), the Seiberg-Witten curve of such theory is an $N$ -fold cover of the sphere with four punctures depicted above.

正如章节"四维中的 Hanany-Witten 膜构型"所讨论的 (见 (123))，该理论的 Seiberg-Witten 曲线是上述带四个 puncture 球面的 $N$ 重覆盖。

In general, if we have the sphere with $N+1$ simple punctures and two maximal punctures, the corresponding theory is described by

一般来说，如果我们有带 $N+1$ 个单 puncture 和两个极大 puncture 的球面，对应的理论可以描述为

$$[N]-\underbrace{(N)-(N)-\cdots-(N)}_{N \text{ gauge groups}}-[N] \tag{142}$$

where $(N)$ denotes an $SU(N)$ gauge group, $[N]$ denotes $N$ flavors of hypermulti-plets in the fundamental representations of the first and last gauge groups, and - denotes the bifundamental hypermultiplets in $\mathrm{SU}(N)\times\mathrm{SU}(N)$ .

其中 $(N)$ 表示一个 $SU(N)$ 规范群，$[N]$ 表示位于首尾规范群基础表示中的 $N$ 味超多重态，-表示位于 $\mathrm{SU}(N)\times\mathrm{SU}(N)$ 中的双基础表示超多重态。

Other theories can be obtained by gluing together tubes and punctured three spheres. The generic class S theory of type $A_{N-1}$ is specified by a Riemann surface $\sum_g$ with punctures with prescribed boundary conditions. The Seiberg-Witten curves of such 4d theories are an $N$ -fold cover of the Riemann surface $\sum_g$ .

其他理论可通过拼接管与带孔三维球面得到。S 类中 $A_{N-1}$ 型的一般理论由带有规定边界条件 puncture 的黎曼曲面 $\sum_g$ 刻画，这类四维理论的西伯格-维滕曲线是黎曼曲面 $\sum_g$ 的 $N$ 重复盖。

## Application I: S-Duality

## 应用 I:S 对偶

Let us consider the $6d\mathcal{N}=(2,0)A_1$ theory on a sphere with four punctures. Recall that in this case there is no distinction between full and simple punctures. We use a complex variable $z$ to parametrize the sphere. We put the punctures at $z=0$ , $z=q, z=1$ , and $z=\infty$ and denote them by $A, B, C$ , and $D$ .

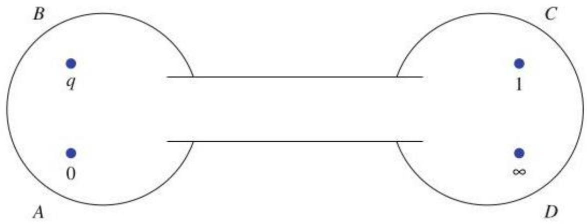
我们考虑四 punctured 球面上的 $6d\mathcal{N}=(2,0)A_1$ 理论。回顾可知，在该情形下满 puncture 与单 puncture 没有区别。我们用复变量 $z$ 参数化该球面，将 puncture 放置在 $z=0$ 、$z=q, z=1$ 和 $z=\infty$ 处，分别记为 $A, B, C$ 和 $D$ 。

In a weakly coupled limit, this can also be described by taking two copies of the $T_2$ theory, namely, $T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_B, \mathrm{SU}(2)_X\right)$ , and $T_2\left(\mathrm{SU}(2)_X, \mathrm{SU}(2)_C, \mathrm{SU}(2)_D\right)$ , and gauging the $\mathrm{SU}(2)_X$ by the 4 $\mathrm{d}\mathcal{N}=2$ vector multiplet with gauge coupling $q$ such that $q\to 0$ . Since the puncture $z=q$ approaches the puncture $z=0$ in this limit, the theory can be described as

在弱耦合极限下，该理论也可以通过如下方式构造: 取两份 $T_2$ 理论，即 $T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_B, \mathrm{SU}(2)_X\right)$ 和 $T_2\left(\mathrm{SU}(2)_X, \mathrm{SU}(2)_C, \mathrm{SU}(2)_D\right)$ ，将 $\mathrm{SU}(2)_X$ 用 4 $\mathrm{d}\mathcal{N}=2$ 矢量多重态做规范，规范耦合常数为 $q$ ，满足 $q\to 0$ 。由于该极限下 puncture $z=q$ 趋近于 puncture $z=0$ ， 该理论可描述为

(143)where the tube can be regarded as very long.

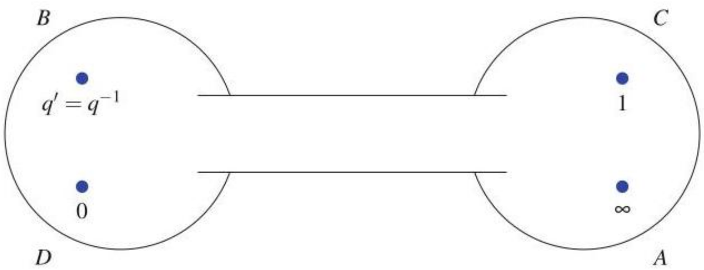
其中管道可视为极长。




Let us consider another limit, namely, $q \to \infty$ , which corresponds to a strongly coupled limit of the theory in question. In order to find a description of this theory, we perform the change of coordinates $z' = z^{-1}$ so that the punctures $A, B, C$ , and $D$ are now at $z' = \infty, q^{-1}, 1, 0$ , respectively. The theory can then be described by

我们考虑另一个极限，即 $q \to \infty$ ，这对应于我们讨论的理论的强耦合极限。为了得到该理论的描述，我们做坐标变换 $z' = z^{-1}$ ，使得 puncture $A, B, C$ 和 $D$ 现在分别位于 $z' = \infty, q^{-1}, 1, 0$ 。此时该理论可描述为

(144)



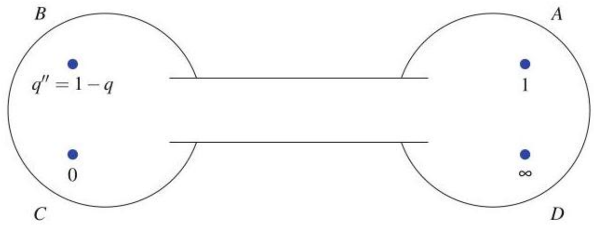

Since $q' \to 0$ as $q \to \infty$ , this can be identified with another weakly coupled theory: a product of $T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_C, \mathrm{SU}(2)_{X'}\right)$ , and $T_2\left(\mathrm{SU}(2)_{X'}, \mathrm{SU}(2)_B, \mathrm{SU}(2)_D\right)$ with the $\mathrm{SU}(2)_{X'}$ gauged by the 4 d$\mathcal{N} = 2$ vector multiplet with gauge coupling $q' = q^{-1}$ .

由于当 $q \to \infty$ 时 $q' \to 0$ 成立，这可以等同于另一个弱耦合理论：$T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_C, \mathrm{SU}(2)_{X'}\right)$ 和 $T_2\left(\mathrm{SU}(2)_{X'}, \mathrm{SU}(2)_B, \mathrm{SU}(2)_D\right)$ 的乘积，其中 $\mathrm{SU}(2)_{X'}$ 被 4 d$\mathcal{N} = 2$ 矢量多重态做规范，规范耦合常数为 $q' = q^{-1}$ 。

We can also consider a limit $q \to 1$ , which corresponds to another strongly coupled regime of the theory. The description of the theory can be realized by taking $z'' = 1 - z$ so that the punctures $A, B, C$ , and $D$ are now at $z'' = 1, 1 - q, 0, \infty$ , respectively. The theory can then be described by

我们还可以考虑极限 $q \to 1$ ，它对应该理论的另一个强耦合区域。通过取 $z'' = 1 - z$ 可以得到该理论的描述，使得 puncture $A, B, C$ 和 $D$ 现在分别位于 $z'' = 1, 1 - q, 0, \infty$ 。此时该理论可描述为

(145)




The above procedure demonstrates a duality that relates the theory at gauge couplings $q, q^{-1}$ and $1-q$ . This is a notion of the $S$ -duality. In summary, we just established the following statement:

上述过程展示了一个联系规范耦合 $q, q^{-1}$ 与 $1-q$ 处理论的对偶。这就是 $S$ 对偶的概念。综上，我们刚刚得到了以下结论:

$$
\begin{aligned}
&\left[T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_B, \mathrm{SU}(2)_X\right) \times T_2\left(\mathrm{SU}(2)_C, \mathrm{SU}(2)_D, \mathrm{SU}(2)_X\right)\right] /_q \mathrm{SU}(2)_X \\
=&\left[T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_C, \mathrm{SU}(2)_{X'}\right) \times T_2\left(\mathrm{SU}(2)_B, \mathrm{SU}(2)_D, \mathrm{SU}(2)_{X'}\right)\right] /_{q^{-1}} \mathrm{SU}(2)_{X'} \\
=&\left[T_2\left(\mathrm{SU}(2)_A, \mathrm{SU}(2)_D, \mathrm{SU}(2)_{X''}\right) \times T_2\left(\mathrm{SU}(2)_B, \mathrm{SU}(2)_C, \mathrm{SU}(2)_{X'}\right)\right] /_{1-q} \mathrm{SU}(2)_{X''} .
\end{aligned}
$$

(146)

## SU(2) Gauge Theory with 4 Flavors

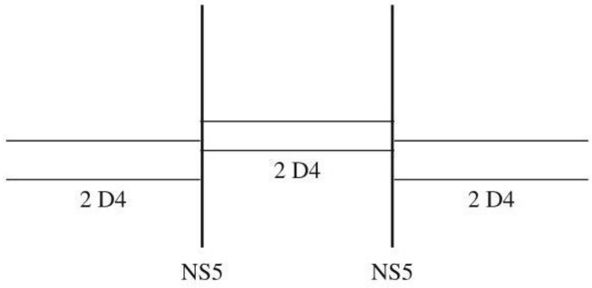
## 含 4 味的 SU(2) 规范理论

The description (143) corresponds to the theory on two M5-branes wrapping a Riemann surface with four punctures. This is the M-theory realization of the following type IIA brane configuration:

描述 (143) 对应两张 M5 膜缠绕一个带四个 puncture 的黎曼曲面得到的理论。这是如下 IIA 型弦论膜构型的 M 理论实现:

(147)where the four punctures correspond to the two NS-branes and the semi-infinite D4- branes and the tube in the middle corresponds to the two finite D4-branes in the middle of the diagram. This is indeed the realization of the SU(2) gauge theory with four flavors of hypermultiplets in the fundamental representation.

其中四个 puncture 对应两张 NS 膜和半无限 D4 膜，中间的管对应图中间的两张有限 D4 膜。这确实是基础表示下含四个超多重态味的 SU(2) 规范理论的实现。




Recall that the description (143) also has an interpretation of taking two copies of the $T_2$ theory and gauging a diagonal SU(2) flavor symmetries corresponding to the punctures of each copy. In order to reconcile this with the SU(2) gauge theory with four flavors, the $T_2$ theory must be a theory of four free hypermultiplets, in such a way that the chiral multiplets transform in the trifundamental representation of the $\mathrm{SU}(2)^3$ manifest flavor symmetry. This confirms the earlier statement we made about the $T_2$ theory.

我们知道，描述 (143) 还可以解释为取两份 $T_2$ 理论，对对应每个拷贝 puncture 的对角 SU(2) 味对称性做规范。为了让这和含四味的 SU(2) 规范理论自洽，$T_2$ 理论必须是四个自由超多重态的理论，且手征多重态按 $\mathrm{SU}(2)^3$ 显式味对称性的三基本表示变换。这就证实了我们之前关于 $T_2$ 理论的结论。

$S$ -duality of the SU(2) gauge theory with four flavors, whose flavor symmetry is Spin(8), has an interesting structure [4, 5]. Suppose that we take the complex mass parameters $m_{1,2,3,4}$ to be associated with each puncture in (143) and that

味对称性为 Spin(8) 的含四味 SU(2) 规范理论的 $S$ 对偶具有有趣的结构 [4,5]。假设我们取复质量参数 $m_{1,2,3,4}$ 对应 (143) 中的每个 puncture，且

$$\pm m_1,\ \pm m_2,\ \pm m_3,\ \pm m_4 \tag{148}$$

correspond to the weight of the vector representation $\mathbf{8}_v$ of Spin(8). The group Spin(8) has an outer automorphism, known as the triality, that permutes the vector $\mathbf{8}_v$ , the spinor $\mathbf{8}_s$ , and the co-spinor $\mathbf{8}_c$ representations. Indeed, $S$ -duality of the SU(2) gauge theory with four flavors is accompanied by the triality of the three duality frames (143), (144), and (145), where in the latter two the mass parameters are given by

对应 Spin(8) 矢量表示 $\mathbf{8}_v$ 的权。Spin(8) 群有一个称为三变性的外自同构，它可以置换矢量 $\mathbf{8}_v$ 、旋量 $\mathbf{8}_s$ 和余旋量 $\mathbf{8}_c$ 表示。确实，含四味 SU(2) 规范理论的 $S$ 对偶伴随着三个对偶框架 (143)、(144) 和 (145) 的三变，后两个框架的质量参数分别由

$$\pm\frac{1}{2}(m_1+m_2+m_3-m_4),\pm\frac{1}{2}(m_1+m_2-m_3+m_4), \tag{149}$$

$$\pm\frac{1}{2}(m_1-m_2+m_3+m_4),\pm\frac{1}{2}(-m_1+m_2+m_3+m_4),$$

and

和

$$\pm\frac{1}{2}(m_1+m_2+m_3+m_4), \pm\frac{1}{2}(m_1+m_2-m_3-m_4), \tag{150}$$

$$\pm\frac{1}{2}(m_1-m_2+m_3-m_4), \pm\frac{1}{2}(m_1-m_2-m_3+m_4).$$

These are indeed the weights of the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations of Spin(8).

给出，它们确实是 Spin(8) 的 $\mathbf{8}_s$ 表示和 $\mathbf{8}_c$ 表示的权。

## Application II: Argyres-Seiberg Duality

## 应用二: 阿吉雷斯-塞伯格对偶

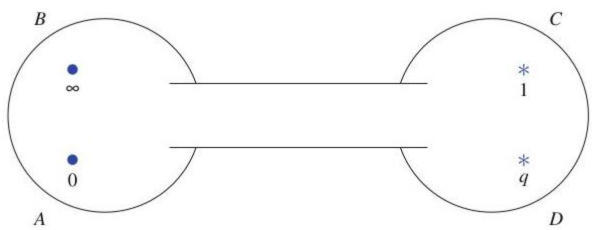
In this section, we discuss a type of duality that generalizes the $S$ -duality described in section ”Application I: S-Duality”. For simplicity, we will study the case of $N=3$ . Let us consider the SU(3) gauge theory with six flavors realized by (141).

本节我们讨论一类推广了“应用一:S 对偶”章节所述 $S$ 对偶的对偶性。为简化讨论，我们研究 $N=3$ 的情况。我们来考察由 (141) 实现的六味 SU(3) 规范理论。

When the coupling $q$ of the SU(3) gauge group is small, this theory admits the description (139). Let us now consider the limit $q\to 1$ , where the theory becomes

当 SU(3) 规范群的耦合 $q$ 很小时，该理论可以用描述 (139) 表示。现在我们来考虑极限 $q\to 1$ ，此时理论变为

(151)




This theory can also be described as follows:

该理论也可以表述如下:

$$\left[T_N\left(\mathrm{SU}(3)_A, \mathrm{SU}(3)_B, X\right)\times \mathrm{S}\left(X, [2,1]_C, [2,1]_D\right)\right]/_{1-q}X \tag{152}$$

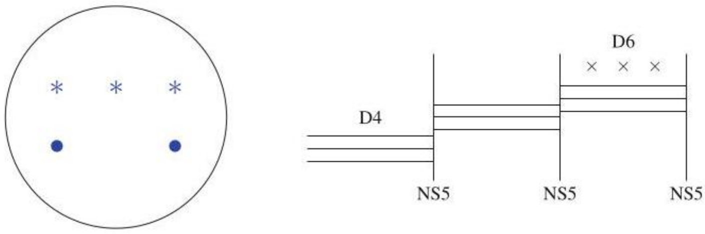
where we denote the theory associated with the sphere on the right by $\mathrm{S}\left(X,[2,1]_C,[2,1]_D\right)$ . The latter needs further analysis. Observe that $\mathrm{S}\left(X,[2,1]_C\ ,[2,1]\,]_D\right)$ can be obtained by partially closing one of the full punctures $[1^3]$ in (137) to a simple puncture [2,1]. Since the theory (137) is a theory of nine free hypermultiplets, it is expected that $\mathbf{S}\left(X,[2,1]_C,[2,1]_D\right)$ is also a theory of free hypermultiplets of smaller amounts.

其中我们将右侧球面对应的理论记为 $\mathrm{S}(X,[2,1]_C,[2,1]_D)$ 。后者需要进一步分析。注意到 ( $\mathrm{S}(X,[2,1]_C$ ，$[2,1]]_D$ ) 可通过将 (137) 中一个满 puncture $[1^3]$ 部分闭合并变为简单 puncture[2,1] 得到。由于理论 (137) 是九个自由超多重态的理论，因此预期 $\mathbf{S}(X,[2,1]_C,[2,1]_D)$ 也是自由超多重态的理论，只是数量更少。

In order to identify $\mathbf{S}(X,[2,1]_C,[2,1]_D)]$ , we proceed as follows: First, we consider the following theory:

为了识别 $\mathbf{S}(X,[2,1]_C,[2,1]_D)]$ ，我们按如下步骤操作: 首先，我们考察如下理论:

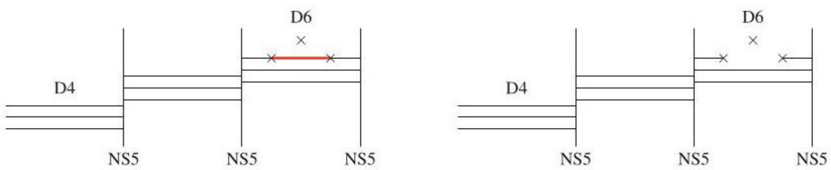
$$[3]-(3)-(3)-[3]$$




(153)

We now partially close the right full puncture to a simple puncture. This can be conveniently realized in the brane system by first bringing two (out of three) D6- branes to cut a D4-branes and then moving the brane segment (highlighted in red below) to infinity in the directions along the D6-branes:

现在我们将右侧的满孔部分闭合成简单孔。这可以在膜系统中方便实现: 先将三个 D6 膜中的两个移去切割一条 D4 膜，再将下方红色高亮的膜段沿 D6 膜方向移到无穷远:




(154)

Applying the Hanany-Witten transition and decoupling the rightmost D6-brane, we obtain the required system:

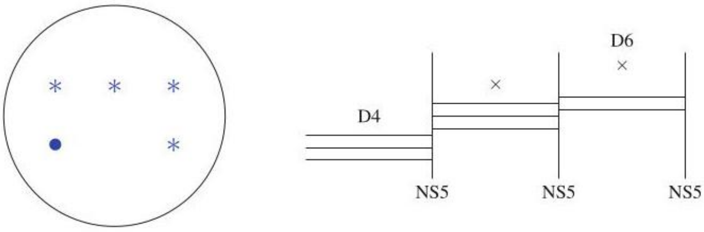
应用哈纳尼-威滕变换并退耦合最右侧的 D6 膜，我们得到所需的系统:

(155)where the brane configuration describes the gauge theory

其中该膜构型描述的规范理论为

$$[3]-(3)-(2)-[1]$$

(156)

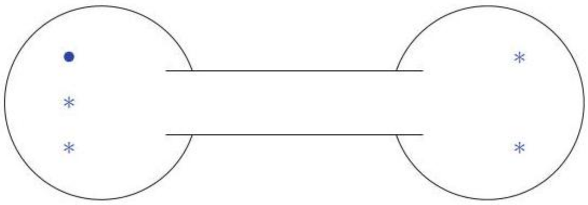



We can then isolate the contribution of $\mathbf{S}\left(X,[2,1]_C,[2,1]_D\right)\big]$ by making the SU(2) gauge group very weakly coupled:

我们可以通过让 SU(2) 规范群的耦合极弱来分离出 $\mathbf{S}\left(X,[2,1]_C,[2,1]_D\right)\big]$ 的贡献:

$$\frac{[3]-(3)-[3]_X\times[2]_X-[1]}{\mathrm{SU}(2)_X}$$

(157)



where the notation highlighted in red means that we gauge the SU(2) subgroup of the SU(3) flavor symmetry. We thus identify $\mathrm{S}\left(X,[2,1]_C,[2,1]_D\right)\big]$ as a doublet of hypermultiplets under a global $\mathrm{SU}(2)_X$ , i.e., $\mathrm{SU}(2)_X$ with one flavor of the hypermultiplet in the fundamental representation.

其中红色高亮的记号表示我们对 SU(3) 味对称性的 SU(2) 子群做规范处理。我们因此将 $\mathrm{S}\left(X,[2,1]_C,[2,1]_D\right)\big]$ 识别为整体 $\mathrm{SU}(2)_X$ 下超多重态的二重态，即带有一个基础表示超多重味的 $\mathrm{SU}(2)_X$ 。

Substituting this back to (152), we obtain the duality proposed by Argyres and Seiberg [59]:

将此结果代回 (152)，我们就得到阿吉雷斯和塞伯格提出的对偶 [59]:

SU(3) gauge theory with six flavors with gauge coupling $q$

规范耦合为 $q$ 的六味 SU(3) 规范理论

$$=\left[\mathrm{T_N}\left(\mathrm{SU}\left(3\right),\mathrm{SU}\left(3\right),\mathrm{SU}(3)_X\right)\times\left(\mathrm{SU}(2)_X\ \text{ with one flavor }\right)\right]/_{1-q}\mathrm{SU}(2)_X\ .$$

(158)

It is interesting to compare the flavor symmetries of the theories on both sides. On the left hand side, there is an SU (6) × U (1) flavor symmetry, where the U (1) factor is identified as the baryonic symmetry. However, only the SU (3) × SU (3) × $\mathrm{U}(1)_{X'} \times \mathrm{U}(1)_F$ is manifest on the right hand side, where $\mathrm{U}(1)_{X'}$ is the commutant of $\mathrm{SU}(2)_X$ in $\mathrm{SU}(3)_X$ and $\mathrm{U}(1)_F$ is the flavor symmetry of the $\mathrm{SU}(2)_X$ gauge theory with one flavor. In order to be consistent with this duality, any two factors of SU(3) in the manifest $\mathrm{SU}(3)^3$ symmetry of the $T_3$ theory, together with a U(1) symmetry, must get enhanced to SU(6). This is only possible if the $T_3$ theory has an $E_6$ flavor symmetry, as we claimed earlier. In fact, the second line of (158) should be interpreted as follows:

比较两边理论的味对称性是很有意思的。左手边存在 SU (6) × U (1) 味对称性，其中 U (1) 因子被识别为重子对称性。但右手边只有 SU (3) × SU (3) × $\mathrm{U}(1)_{X'} \times \mathrm{U}(1)_F$ 是明显的，其中 $\mathrm{U}(1)_{X'}$ 是 $\mathrm{SU}(2)_X$ 在 $\mathrm{SU}(3)_X$ 中的交换子，且 $\mathrm{U}(1)_F$ 是单味 $\mathrm{SU}(2)_X$ 规范理论的味对称性。为了与该对偶自洽，$T_3$ 理论的明显 $\mathrm{SU}(3)^3$ 对称性中任意两个 SU(3) 因子，加上一个 U(1) 对称性，必须增强为 SU(6)。正如我们之前所说，这只有当 $T_3$ 理论具有 $E_6$ 味对称性时才成立。实际上，(158) 的第二行应按如下解释:

1. We first decompose the $E_6$ flavor symmetry of $T_3$ into a maximal subgroup SU (6) × $\mathrm{SU}(2)_X$ .

1. 我们首先将 $T_3$ 的 $E_6$ 味对称性分解为极大子群 SU (6) × $\mathrm{SU}(2)_X$ 。

2. We then couple it to a doublet of hypermultiplets in $\mathrm{SU}(2)_X$ .

2. 然后我们将其与 $\mathrm{SU}(2)_X$ 中的超多重子二重态耦合。

3. We gauge the diagonal $\mathrm{SU}(2)_X$ symmetry with coupling $1-q$ .

3. 我们用耦合常数 $1-q$ 规范对角 $\mathrm{SU}(2)_X$ 对称性。

We remark that the $\mathrm{SU}(2)_X$ gauge theory with a doublet of hypermultiplets (i.e., with one flavor) has a $\mathrm{U}(1)_F$ flavor symmetry. This is identified with the U(1) baryonic symmetry of the theory on the first line of (158). The ungauged SU(6) flavor symmetry in the above discussion is identified with the SU(6) flavor symmetry of the theory on the first line of (158).

我们注意到，带有超多重子二重态 (即单味) 的 $\mathrm{SU}(2)_X$ 规范理论具有 $\mathrm{U}(1)_F$ 味对称性。这对应于 (158) 第一行中理论的 U(1) 重子对称性。上述讨论中未被规范的 SU(6) 味对称性对应于 (158) 第一行中理论的 SU(6) 味对称性。

As an application of the Argyres-Seiberg duality (158), it is possible to determine the Higgs branch of the $T_3$ theory [60, 61] . This turns out to be the centered moduli space of one $E_6$ instanton on $\mathbb{R}^4$ . Even though the ADHM construction of the $E_6$ instanton moduli space is not available, it can be shown using (158) that the Higgs branch of the $T_3$ theory is a closure of the minimal nilpotent orbit of $E_6$ . It was shown by Kronheimer [62] that this hyperKähler singularity is precisely the expected moduli space of one $E_6$ instanton.

作为 Argyres-Seiberg 对偶 (158) 的一个应用，我们可以确定 $T_3$ 理论 [60, 61] 的希格斯分支。结果表明它是 $\mathbb{R}^4$ 上单个 $E_6$ 瞬子的中心模空间。尽管 $E_6$ 瞬子模空间的 ADHM 构造尚不存在，利用 (158) 可以证明 $T_3$ 理论的希格斯分支是 $E_6$ 极小幂零轨道的闭包。Kronheimer[62] 已证明该超凯勒奇点正好是单个 $E_6$ 瞬子的预期模空间。

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